

A COMPARISON OF THE APPLICABILITY AND EFFECTIVENESS
OF ANOVA WITH MANOVA FOR USE IN THE OPERATIONAL
EVALUATION OF COMMAND AND CONTROL SYSTEMS

A THESIS

Presented to
The Faculty of the Division
of Graduate Studies


By
Thomas Nelson Burnette, Jr.

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Operations Research

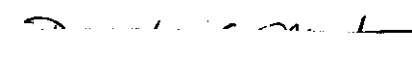
Georgia Institute of Technology
May, 1975

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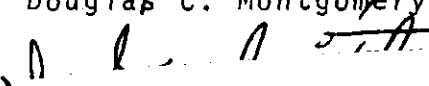
Approved:



Leslie G. Callahan, Jr. Chairman



Douglas C. Montgomery



I. Gayden Thompson

Date approved by Chairman: 15 May 75

ACKNOWLEDGMENTS

I would like to express my appreciation to all those who assisted me in preparing this thesis. I am particularly grateful to Major General Elmer R. Ochs, Commander, U. S. Army Operational Test and Evaluation Agency, for the contractual support which made this research possible.

My sincere appreciation goes to Dr. Leslie G. Callahan, Jr., chairman of my committee, for his guidance, patience, and critical evaluations; to Dr. Douglas C. Montgomery, from whom I learned the mysteries of statistics, for his assistance in the research and writing; and to Dr. I. Gayden Thompson for his guidance in the simulation aspects of the research.

To the United States Army, I extend a special expression of gratitude for the opportunity to attend graduate school and the opportunity to conduct this research.

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SUMMARY

This research addresses the problem of developing a usable methodology with which to compare the effectiveness of univariate analysis of variance (ANOVA) with multivariate analysis of variance (MANOVA), on the basis of powers of the tests, for use in comparative operational testing. The scope of the research was limited by considering only completely-crossed designs; two-factor, fixed-effects models; equal cell sample sizes; and no effect due to operators. In addition, it was assumed that an estimate of the multi-response correlation matrix is available.

In order to make analysis possible, a procedure with which to determine the power of the MANOVA test was required. The MANOVA power function is not known in a closed or usable form; consequently, a Monte Carlo procedure was devised to determine the power of the MANOVA test. The maximum likelihood form of the MANOVA test statistic was utilized due to its ease of computation and attendant power considerations.

Three MANOVA power criteria were developed which are extensions of the ANOVA power criteria. The following general results were found to hold for the MANOVA power function:

1. Power is a decreasing function of the dimension of the multiresponse.

2. Power is an increasing function of the size departure from the null hypothesis.
3. Power is an increasing function of sample size.
4. Power is an increasing function of the probability of Type I error.
5. Power is an increasing function of $-\log |P|$, where P is the correlation matrix of the multiresponse.

A methodology was developed, using the MANOVA Monte Carlo power procedure, for comparing the effectiveness of ANOVA with MANOVA for a correlated set of responses, under the assumption that the system in question satisfies the assumptions required for both techniques. An example of the use of the methodology is included. A FORTRAN IV listing of the MANOVA Monte Carlo power program is also included.

CHAPTER I

INTRODUCTION

Background

Department of the Army Major Systems Acquisition Procedure

The procedure by which the Department of Defense acquires a major defense system is well structured and reinforced with safeguards to prevent the acquisition of unnecessary or unsatisfactory systems. The procedure utilized by the Department of the Army for major systems acquisition complements and closely parallels the procedure employed by the Department of Defense. Highly regulated measures are utilized to insure that only those systems for which a valid need exists are acquired by the Department of Defense. The measures are described at some length in various Department of Defense directives (8,21,22).

Prior to full production and deployment, a major system will pass through three previous phases after it has been determined by the Army Staff that a valid requirement exists for the proposed system. The first phase is the conceptual development phase during which the systems hardware is in an experimental prototype configuration. The second phase is the full scale development phase during which the systems hardware is in an engineering development

prototype configuration. The third phase is the full scale development phase during which the systems hardware is in a production prototype configuration (8).

The Secretary of Defense must give his approval before a major system may transition from one phase to the next. A permanent advisory body, The Defense Systems Acquisition Review Council (DSARC), is in being to provide information and recommendations to the Secretary of Defense whenever program decisions become necessary. Principal members of the DSARC include the Deputy Secretary of Defense and those Assistant Secretaries of Defense for systems within their areas of responsibility. A scheduled meeting of the DSARC precedes the Secretary of Defense's decision whether to proceed with system development at each phase transition point.

At each phase transition point the Secretary of Defense may opt to terminate the system, permit the system to proceed to the next phase, or retain the system in its present phase for remedial action (22).

Within the Department of the Army there exists a procedure which parallels and precedes the DSARC procedure. Another permanent advisory body, the Army Systems Acquisition Review Council (ASARC), is in being to provide the DSARC with the Army's recommendation at each phase in the acquisition process. The ASARC is a high-level body chaired by the Vice

Chief of Staff of the Army. Its principal members include the Commander of the U. S. Army Materiel Command, the Commander of the U. S. Army Training and Doctrine Command, the Chief of Research, Development, and Acquisition, and various Assistant Secretaries of the Army. Scheduled meetings of the ASARC precede those of the DSARC.

It is important to remember that the final decision at each phase transition point rests with the Secretary of Defense. This is in keeping with the principle of civilian control of the military acquisition process (8).

Requirement for Testing

Testing of a major system is conducted to determine whether the system meets its technical and operational requirements. For the purposes of acquisition, testing is grouped into two categories: Developmental Testing (DT) and Operational Testing (OT). DT and OT differ in their objectives. DT is conducted in order to determine whether the engineering design and development process is complete, to determine whether the design risks have been minimized, and to determine whether the system will meet its specifications. OT is conducted to estimate the system's military worth in comparison with competitor systems, to estimate its operational effectiveness and suitability in its environment, and to determine whether the system requires modification (8).

Normally three distinct DT and OT are conducted for each major system. One DT and OT precedes each of the three scheduled meetings of the ASARC and DSARC. Figure 1 depicts the relationship between the DT, OT, ASARC, and DSARC.

Results of the DT and OT are reported directly to the ASARC for inclusion in its recommendation to the DSARC. The DT and OT are required to be evaluated independently of each other (8). This research will be concerned with OT only.

Operational Testing

Operational testing is conducted by an organization independent of the developing/procuring and using organizations. OT is accomplished using typical user/operators, crews, or units in as realistic an operational environment as possible. The OT are conducted in such a manner as to provide the necessary data to estimate:

1. The military utility, operational effectiveness, and operational suitability of the system.
2. The system's desirability, considering systems already in service and other competing developmental systems, and the system's operational benefits and burdens from the user's viewpoint.
3. The need for modification of the system.
4. The adequacy of doctrine, organization, operating techniques, tactics, and training for system employment.
5. The adequacy of maintenance support for the system.

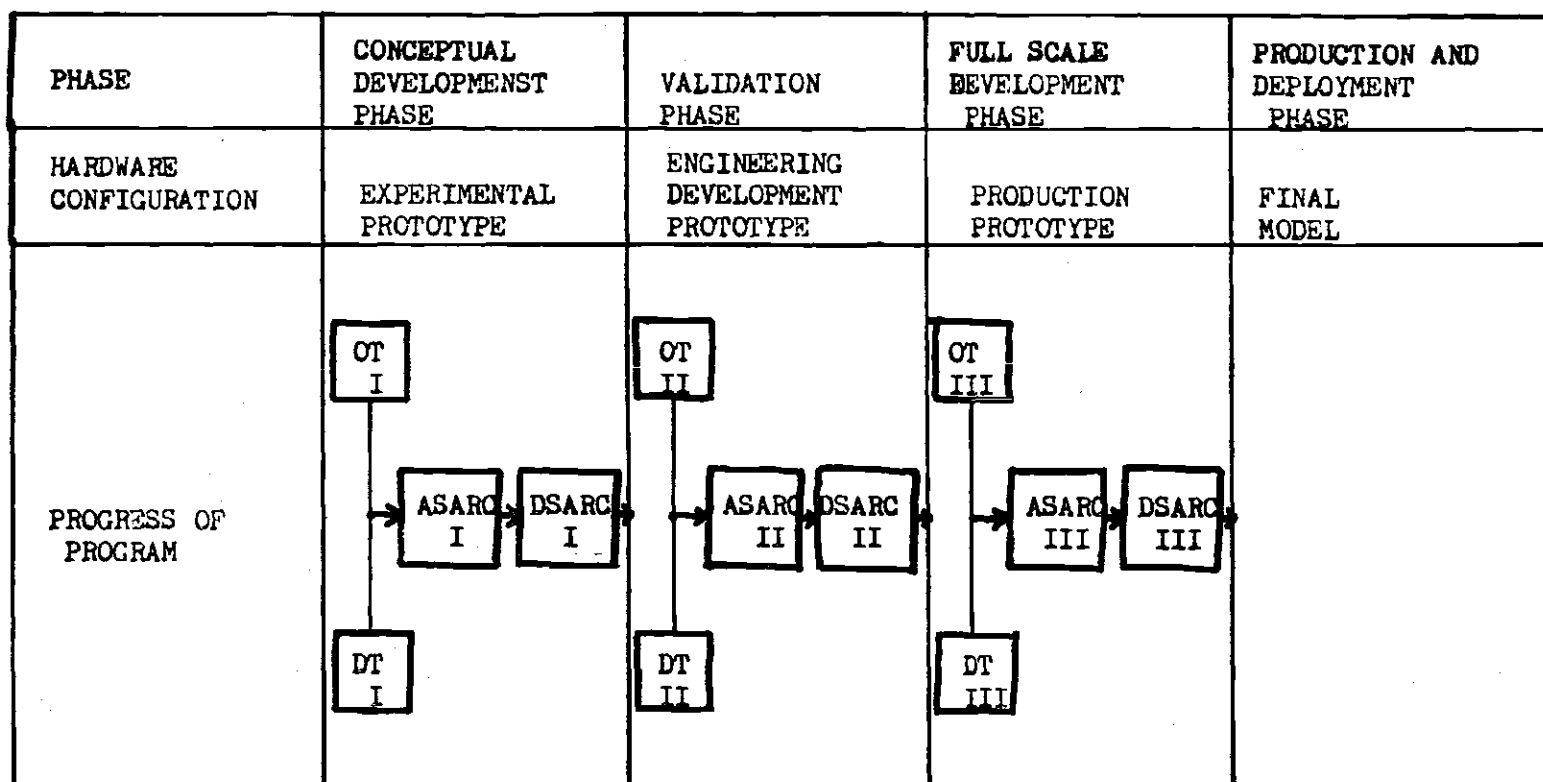


Figure 1. Defense Systems Acquisition Process

6. The system's performance in a countermeasures environment.

The operational testor prepares an independent evaluation after each OT and reports the results directly to the ASARC. Throughout the process the OT will emphasize the comparative evaluation of the new system with existing systems and competitor developmental systems. The U. S. Army Test and Evaluation Agency is designated as the agency responsible for OT on major defense systems (6, 7).

Command and Control Systems

During the last decade the U. S. Army has expended a great deal of time and money to develop and deploy a number of sophisticated tactical command and control systems. For the purpose of this research a tactical command and control system is defined as an arrangement of personnel, facilities, and the means for information acquisition, processing, and dissemination employed by a commander in planning, directing and controlling tactical operations. Recent tactical command and control systems under development include the Tactical Operations Systems (TOS), a division-level command and control system; the TSQ-73, an air defense command and control system; and TACFIRE, an artillery fire control and fire support command and control system.

Measures of effectiveness employed in the operational test and evaluation of tactical command and control systems

vary; however, due to the nature of these systems, rarely will the measures of effectiveness be independent of each other (55). For instance, in a division-level command and control system such measures of effectiveness as fraction of available planning time passed to subordinate echelons and time required to prepare staff actions are highly correlated (55).

It is U. S. Army policy that preference be given to evolutionary development of existing systems. Consequently, developmental command and control systems will be tested and evaluated in comparison with existing command and control systems as a rule. The U. S. Army Training and Doctrine Command develops and maintains standard scenarios for use in the OT of command and control systems (7). The OT are in effect designed experiments in which two, but rarely more than three, systems are compared while functioning in their operating environment.

The appropriate statistical technique to utilize in the OT of command and control systems appears to be the analysis of variance. Historically, Army testers have relied solely upon univariate statistical techniques as the vehicle for comparing systems. However, in command and control systems many of the measures of effectiveness appear to be highly correlated. This correlation of variables suggests that some form of multivariate analysis may be

appropriate. For systems in which some groups of variables are correlated and some groups are not, some combination of univariate and multivariate techniques may be appropriate. A promising area for research appears to exist in developing a methodology for the comparison of the applicability and effectiveness of univariate analysis of variance (ANOVA) with multivariate analysis of variance (MANOVA) for use in the comparative operational test and evaluation of tactical command and control systems.

Objective, Procedure and Scope

The primary objective of this research is to develop a usable methodology with which to compare the applicability and effectiveness of univariate analysis of variance (ANOVA) with multivariate analysis of variance (MANOVA) for use in the comparative operational test and evaluation of alternative tactical command and control systems. The investigation will consist of a review of ANOVA and MANOVA techniques, the synthesis of a methodology for comparing the applicability and effectiveness of ANOVA with MANOVA, a demonstration of the methodology, and considerations for its application. The research will address the following factors:

1. The assumptions required for each technique.
2. The effects of departures from the required assumptions.
3. The powers of the tests versus correlation,

sample size, and probability of Type I error.

4. The validity of probability statements concerning systems parameters.

The scope of this research will be limited by three realistic assumptions. First, due to the prototype configurations of the systems hardware and the nature of the specified standard scenarios used in OT, only the fixed effects model of ANOVA and MANOVA will be considered appropriate. Second, in keeping with the systems approach, equal cell sample sizes only will be considered appropriate for each technique. Third, due to the interactive nature of command and control systems operations and the prohibitively high costs involved in training more than one command and staff group to operate each alternative command and control system, operators of the alternative systems will not be considered a factor. In addition to the three assumptions listed above, in order to limit the programming which will be required, only two-factor, completely crossed designs will be considered.

CHAPTER II

REVIEW OF APPLICABLE STATISTICAL RESULTS AND TECHNIQUES

Introduction

This chapter is intended as a brief review of the statistical results and techniques which may be of use in developing a methodology for use in comparing the applicability and effectiveness of ANOVA with MANOVA. We will not trace the historical development of ANOVA and MANOVA; this chapter is not meant to honor those who have contributed to this branch of statistics, nor is it intended to be a tutorial. Rather, we will review a number of significant results and techniques which may be of use to us, to include the univariate analysis of variance, ANOVA, the multivariate analysis of variance, MANOVA, correlation analysis, and the generation of multivariate normal random vectors.

Univariate Analysis of Variance

The appropriate univariate statistical model for use in comparing several systems is the analysis of variance, ANOVA. For the general fixed-effects case we would have m different factors with different levels of each factor. There would be a levels of factor A, b levels of factor B, and so on, with m levels of factor M. If we consider a completely crossed factorial experiment with n equal

observations for each unique combination of factors (cell) we would have $(nab...m)$ total observations. We will review the model and assumptions for the two-factor case; however, the model and assumptions are easily extended for the general case. We will only consider completely crossed designs and fixed-effects models in the course of this research.

Model and Required Assumptions

The two-factor fixed-effects model ANOVA is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \quad (2.1)$$

$$i = 1, \dots, a$$

$$j = 1, \dots, b$$

$$k = 1, \dots, n$$

μ is the mean effect common to all observations, α_i is the effect due to level i of factor A; β_j is the effect due to level j of factor B. γ_{ij} is the effect due to interaction of level i of factor A with level j of factor B. e_{ijk} is the effect due to random error in the k th observation with factor A at level i and factor B at level j (49,33).

For the purpose of estimation, inference, and hypothesis testing, we must make a number of assumptions. Concerning the effects due to the levels of the factors and interaction we assume:

$$\sum_{i=1}^a \alpha_i = 0 = \sum_{j=1}^b \beta_j \quad (2.2)$$

and

$$\sum_{i=1}^a \gamma_{ij} = 0 \quad j = 1, \dots, b \quad (2.3)$$

and

$$\sum_{j=1}^b \gamma_{ij} = 0 \quad i = 1, \dots, a \quad (2.4)$$

Concerning the random error we assume:

$$e_{ijk} \text{ are distributed independently } N(0, \sigma^2) \quad (2-5)$$

While the model is linear, it is not purely additive in that interaction terms exist (49).

Hypothesis Testing

Appropriate hypotheses which we might want to test include:

H_{10} : No effect due to factor A or $\alpha_i = 0, i=1, \dots, a$
against

H_{11} : Not H_{10}

H_{20} : No effect due to factor B or $\beta_j = 0, j=1, \dots, b$
against

H_{21} : Not H_{20}

H_{30} : No effect due to interaction or $\gamma_{ij} = 0, i=1, \dots, a, j=1, \dots, b$
against

H_{31} : Not H_{30}

The ANOVA procedure consists of partitioning the total variation (sums of squares) in the observations into components due to the main effects, the interaction, and the random error. For the two-factor model the partition is:

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E \quad (2.6)$$

Computational formulae for the above sums of squares partition are well known and will not be given here (49, 33).

An appropriate test statistic for use in ANOVA hypothesis testing is based upon the F distribution. Under the null hypothesis, say H_{10} : $\alpha_i = 0$, $i=1, \dots, a$, we would reject H_{10} with confidence $(1 - \alpha)$ if

$$F_0 = \frac{SS_A/(a-1)}{SS_E/ab(n-1)} > F_{\alpha, a-1, ab(n-1)} \quad (2.7)$$

where $F_{\alpha, a-1, ab(n-1)}$ is the upper $(1-\alpha)$ percentage point of the F distribution with $a-1$ numerator degrees of freedom and $ab(n-1)$ denominator degrees of freedom (33). Quantities such as $SS_A/(a-1)$, a sum of squares divided by its degrees of freedom, are called mean squares; in the case of factor A it would be written MS_A . Similar tests are used for the other hypotheses (33).

The procedure to follow in ANOVA hypothesis testing is to test for interaction effect first. If we fail to

reject the hypothesis of no interaction, then we may test the hypotheses on the main effects. However, if we reject the hypothesis of no interaction effect, then the main effects will be masked. When interaction is present we may employ a procedure developed by Tukey to test for main effects in the presence of interaction, known as "one degree of freedom for non-additivity" (16, 49, 53).

Power of the Analysis of Variance

In addition to α , the probability of rejecting the hypothesis given it is true, we are interested in the power of the test; the probability of rejecting the hypothesis given it is false. In order to find the power of the ANOVA test, it is necessary to determine the distribution of the test statistic, F_0 , under the alternative hypothesis. Graybill, Scheffe', and others have shown that under the alternative hypothesis F_0 is distributed as a non-central F distribution (30, 49).

For example, consider testing the hypothesis of no effect due to factor A. In order to determine the power of the F test, we would need to determine:

$$\begin{aligned} &P\{F_0 > F_{\alpha, v_1, v_2} | H_1 \text{ is true}\} \\ &= P\{F'_{v_1, v_2, \lambda} > F_{\alpha, v_1, v_2}\} \end{aligned}$$

where $F'_{v_1, v_2, \lambda}$ is the non-central F distribution with $v_1 = a - 1$,

$$v_2 = ab(n-1), \text{ and } \lambda = \frac{n \sum_{i=1}^a \alpha_i^2}{2\sigma^2} \quad (33).$$

Rather than using tables of the non-central F distribution, we may use more convenient charts constructed by Pearson and Hartley. These charts plot the probability of Type II error (1 - power) for various v_1 , v_2 , α , and parameter ϕ , where for the case of H_{10}

$$\phi^2 = \frac{2}{a} \lambda \quad (2.8)$$

Since σ^2 will seldom be known we work with ratios of $\frac{\sum_{i=1}^a \alpha_i^2}{\sigma^2}$ we desire to detect (42).

Estimation and Simultaneous Inference

Unbiased, least-squares estimators of the ANOVA model parameters are available. These estimators are readily yielded from the calculations necessary to test hypotheses. In dot notation

$$\hat{\mu} = \frac{y_{\cdot\cdot\cdot}}{abn} = \bar{y}_{\cdot\cdot\cdot} \quad (2.9)$$

$$\hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot} \quad i = 1, \dots, a \quad (2.10)$$

$$\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot} \quad j = 1, \dots, b \quad (2.11)$$

$$\hat{\gamma}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot} \quad (2.12)$$

$$i = 1, \dots, a; \quad j = 1, \dots, b$$

An unbiased estimator of σ^2 is $\hat{\sigma}^2 = MS_E$ (33).

We will seldom know in advance of a test which factor effects may be significant; therefore, we would like to have a procedure by which we could conduct some form of simultaneous multiple comparison of levels should we reject the null hypothesis in the ANOVA test. We would like to know which levels of the factor under consideration caused the rejection of the hypothesis. Four simultaneous comparison procedures exist which may be applied to this problem. One procedure developed by Tukey is useful in comparing pairs of factor levels, Scheffe' has developed a procedure involving F projections which is more general in nature and can be used for contrasts involving more than two levels of a factor. The other two procedures are similar in nature. One of these two procedures is due to Newman and Keuls and permits simultaneous comparison of pairs of factor levels. The other procedure is due to Duncan and is an extension of the procedure of Newman and Keuls (38).

If we are interested in making more than pairwise comparisons between factor levels, we would use Scheffe's procedure. For pairwise comparisons of factor levels we could use either of the other three procedures. Of these three pairwise procedures, Tukey's procedure is the most conservative, Duncan's is the most liberal, and the Newman-Keuls procedure lies somewhere between the two. The proce-

dures will not be detailed here. An excellent discussion, development, and comparison of the procedures can be found in Miller (38).

Effects of Departures from the Assumptions

We shall consider the effects of violations of the following three assumptions required in the ANOVA:

1. Normality of the random errors.
2. Equality of the error variance.
3. Statistical independence of the errors.

We shall consider these three violations one at a time for the fixed-effects model. Our primary interest lies in the effect of the violations on the probability of Type I error and the power of the test in hypothesis testing.

Researchers, including Pearson and Box, have found that violation of normality of the errors has slight effect on inferences concerning means (12,41). In the fixed-effects ANOVA we are concerned with inferences on means. More recent work by Tiku has clearly demonstrated for the fixed-effects ANOVA that nonnormality has very little effect on either the probability of Type I error or the power of the test (52).

Concerning the violation of equality of error variance, it has been determined that so long as equal sample sizes are taken in each cell, the effect on the probability of Type I error and the power of the test is slight. This validates our assumption that we will only consider equal sample sizes. However, for unequal sample sizes per cell,

the effect on the probability of Type I error and the power of the test can be considerable (13, 49). The best insurance against unequal error variance appears to be equal sample sizes in each cell.

Of the three violations which we consider, correlation of the errors is by far the most serious. Any correlation, especially serial correlation, can have an extremely pernicious effect on the probability of Type I error and the power of the test. The only known counter to correlation can be applied to serial correlation. It consists of estimating the serial correlation coefficient and attempting to estimate its effect (13). It is clear that ANOVA is an inappropriate model when dealing with correlated observations.

Multivariate Analysis of Variance

Model and Required Assumptions

The appropriate multivariate model for use in comparing several multiresponse systems is the multivariate analysis of variance (MANOVA). For the general fixed effects case we have m different factors, with different levels of each factor, and each level of a factor constituting a $p \times 1$ -dimensional vector of the multiresponse. For a completely crossed factorial experiment with n equal observations per unique combination of the factors (cell), there would be a levels of factor A, b levels of factor B, and so on, with m levels of factor M. In all there would be $(ab...mn)$ total

observations. We will review the model and assumptions for the two factor fixed-effects case; however, the model and assumptions can be easily extended for the general case.

The two-factor fixed-effects model is

$$\begin{matrix} \chi_{ijk} & = & \mu & + & \alpha_i & + & \beta_j & + & \gamma_{ij} & + & e_{ijk} \\ (p \times 1) & & (p \times 1) & & (p \times 1) & & (p \times 1) & & (p \times 1) & & (p \times 1) \end{matrix} \quad (2.13)$$

$$i=1, \dots, a; \quad j=1, \dots, b; \quad k=1, \dots, n$$

Here the vector μ is the effect common to all observations. The vector α_i is the effect due to level i of factor A, the vector β_j is the effect due to level j of factor B, and the vector γ_{ij} is the interaction effect due to level i of factor A and level j of factor B. The vector e_{ijk} is the effect due to random error with factor A at level i and factor B at level j on the k th observation (43,46).

The purposes of estimation, inference, and hypothesis testing, we must make several assumptions. Concerning the effects due to the levels of factors and interaction we assume:

$$\sum_{i=1}^a \alpha_i = \phi = \sum_{j=1}^b \beta_j \quad (2.14)$$

$$\sum_{i=1}^a \gamma_{ij} = \phi \quad j=1, \dots, b \quad (2.15)$$

$$\sum_{j=1}^b \gamma_{ij} = \phi \quad i=1, \dots, a \quad (2.16)$$

Concerning the random error we assume e_{ijk} are independently distributed $N(\phi, \Sigma)$, $\Sigma > \phi$ (43,46).

Hypothesis Testing

Appropriate hypotheses we might want to test include:

H_{10} : No effect due to factor A or $\alpha_i = \phi$ $i=1, \dots, a$

against

H_{11} : Not H_{10}

H_{20} : No effect due to factor B or $\beta_j = \phi$ $j=1, \dots, b$

against

H_{21} : Not H_{20}

and

H_{30} : No effect due to interaction or $\gamma_{ij} = \phi$ $i=1, \dots, a$
 $j=1, \dots, b$

against

H_{31} : Not H_{30}

Unlike the ANOVA, there exists no single hypothesis test criterion for the MANOVA. There are three widely used alternative MANOVA hypothesis test criteria. They are the likelihood ratio criterion, the trace criterion, and the largest characteristic root criterion (39). The likelihood ratio criterion is generally preferred due to its ease of calculation and attendant power considerations (48). We will use only the likelihood ratio criterion. Its use requires one additional assumption for the two-factor case, the dimension of the response, $p \leq ab(n-1)$ (43).

The MANOVA hypothesis testing procedure consists of

partitioning the total variation in the observations in a manner similar to the ANOVA partition. Computational formulae will not be presented here; however, instead, we will define the following $p \times p$ matrices:

\underline{E} - matrix of error sums of squares and cross products.

$\underline{H1}$ - matrix of factor A sums of squares and cross products.

$\underline{H2}$ - matrix of factor B sums of squares and cross products.

$\underline{H3}$ - matrix of interaction sums of squares and cross products.

The likelihood ratio test for $H_{10}: \alpha_i = \phi$ is to reject H_{10} if

$$\frac{|\underline{E}|}{|\underline{E} + \underline{H1}|} < \text{constant} \quad (2.17)$$

under H_{10}

$$\mathcal{L} \left\{ \frac{|\underline{E}|}{|\underline{E} + \underline{H1}|} \right\} = \mathcal{L} \{ U_{p,q1,n} \} \quad (2.18)$$

where p is the dimension of the response, $q1 = a-1$, and $n = ab(n-1)$. Hence we reject H_{10} if the test statistic is less than $U_{p,q1,n}$. The appropriate values of U are found using a second order χ^2 approximation developed by Box (11, 43). The test statistics for H_{20} and H_{30} are similar.

As in the ANOVA we would test the hypothesis of no interaction first. If we fail to reject this hypothesis, we would then test the hypotheses on the main effects. There exists no multivariate counterpart of Tukey's "one degree of freedom for non-additivity"; therefore, if we reject the hypothesis of no interaction, we must rely on simpler comparison techniques to determine whether main effects are significant (43).

Power of the Multivariate Analysis of Variance

The power functions for the MANOVA test criteria are not known in closed form. The noncentral distributions of the largest characteristic root and likelihood ratio statistics have been studied in some detail recently; however, to date research has not yielded a usable power function for the MANOVA tests. It has been shown by Roy, Mikhail, and others that the MANOVA power is a monotonically increasing function of the noncentrality parameters of the criteria distributions (47). Using Monte Carlo methods, Gnanadesikan showed that the MANOVA test power is monotonically decreasing with increasing dimension of the response, p , and is monotonically increasing with increasing probability of Type I error. Gnanadesikan also found that in general the likelihood ratio criterion possesses better power characteristics than either of the other two criteria (48). The lack of a usable power function seems to be a major drawback when using the MANOVA.

Estimation and Simultaneous Inference

Unbiased least-squares estimators for the MANOVA parameters are available. As in the ANOVA these estimators are readily yielded from the calculations necessary to test hypotheses. In dot notation

$$\hat{\underline{\mu}}_{(p \times 1)} = \frac{\underline{y}_{\cdot \cdot \cdot}}{abn} = \bar{\underline{y}}_{\cdot \cdot \cdot} \quad (2.19)$$

$$\hat{\underline{\alpha}}_{(p \times 1)} = \bar{\underline{y}}_{i \cdot \cdot} - \bar{\underline{y}}_{\cdot \cdot \cdot} \quad i=1, \dots, a \quad (2.20)$$

$$\hat{\underline{\beta}}_{(p \times 1)} = \bar{\underline{y}}_{\cdot j \cdot} - \bar{\underline{y}}_{\cdot \cdot \cdot} \quad j=1, \dots, b \quad (2.21)$$

$$\hat{\underline{\gamma}}_{(p \times 1)} = \bar{\underline{y}}_{ij \cdot} - \bar{\underline{y}}_{i \cdot \cdot} - \bar{\underline{y}}_{\cdot j \cdot} + \bar{\underline{y}}_{\cdot \cdot \cdot} \quad (2.22)$$

$i=1, \dots, a; j=1, \dots, b.$

An unbiased maximum likelihood estimator of the error covariance matrix is

$$\hat{\underline{\Sigma}}_{(p \times p)} = \frac{1}{ab(n-1)} \underline{E} \quad (2.23)$$

where \underline{E} is the $p \times p$ matrix of error sums of squares and cross products (24).

A variety of techniques are available for simultaneous multiple comparison of levels of an effect should we reject the hypothesis of no effect in the MANOVA. Gabriel has found that so long as we are interested in contrastwise simultaneous inference on all linear combinations of levels,

including pairwise comparison, the procedure utilizing the union-intersection approach is the best from the standpoint of the probability of Type I error and power of the test (27). Confidence intervals based upon the largest characteristic root criterion are formed; in the event that an interval contains zero, the contrast under consideration is considered to be zero (46).

For example, consider the two-factor model we have used. Let $a'_h = (a_1, a_2, \dots, a_p)$ be a nonnull vector for defining contrasts between levels of a factor. Let factor A be the row effect and factor B be the column effect. The 100 $(1 - \alpha)$ percent confidence intervals for the linear compound of differences in the l th and m th row effects (levels of factor A) are given by:

$$\begin{aligned} \sum_{h=1}^p a_h (\bar{x}_{l.h} - \bar{x}_{m.h}) - \sqrt{\frac{2x_{\alpha} a' E a}{bn(1-x_{\alpha})}} &\leq \sum_{h=1}^p a_h (\alpha_{lh} - \alpha_{mh}) \\ &\leq \sum_{h=1}^p a_h (\bar{x}_{l.h} - \bar{x}_{m.h}) + \sqrt{\frac{2x_{\alpha} a' E a}{bn(1-x_{\alpha})}} \end{aligned} \quad (2.24)$$

where $\bar{x}_{l.h}$ is the h th response mean for row level l , E is the matrix of error sums of squares and cross products, and x_{α} is the value from the Heck chart of the largest characteristic root distribution with parameters α , $s = \min(a-1, p)$, $m = (|a-1-p|-1)/2$, and $n = (ab(n-1)-p-1)/2$. We require that the interaction terms be zero; otherwise, comparisons are

meaningless. A similar procedure exists for column levels (43).

Effects of Departures from Assumptions

Unfortunately, very little research has been conducted to determine the effects of departures from the MANOVA assumptions. As with the ANOVA, we are concerned with violations of the following assumptions:

1. Multivariate normality of the errors.
2. Equality of covariance matrices.
3. Statistical independence of the errors.

That research in this area which has been conducted has addressed itself to large sample or asymptotic results. Small sample results appear to be extremely difficult, if not impossible, to derive.

Ito found that for large sample sizes that the violation of multivariate normality of the errors has little effect when testing hypotheses concerning means (34). This result reflects the robustness of fixed-effects ANOVA to nonnormality. It appears that fixed-effects MANOVA tests are robust to violations of multivariate normality. Research has yet to yield general results concerning the robustness of MANOVA tests to unequal covariance matrices and correlation. The effects of departures from MANOVA assumptions on the probability of Type I error and the power of the test are, in fact, unknown.

Correlation Analysis

Simple Correlation

In order to determine whether multivariate statistical analysis is appropriate, we need to examine the correlation structure of the responses. The most elementary analysis of correlation structure involves the simple correlation coefficient, ρ . Let y_1, y_2, \dots, y_n be n independent observations on a p -dimensional random vector Y . The covariance between the i th and j th components of Y , Y^i and Y^j , is defined as

$$\sigma_{ij} = \text{cov}(Y^i, Y^j) = E[(Y^i - EY^i)(Y^j - EY^j)] \quad (2.25)$$

where σ_{ii} denotes the variance of Y^i . The $p \times p$ matrix of population covariances is defined as

$$\Sigma = (\sigma_{ij}) \quad (2.26)$$

The correlation coefficient between Y^i and Y^j is defined as

$$\rho_{ij} = \frac{\sigma_{ij}}{(\sigma_{ii}\sigma_{jj})^{1/2}} \quad -1 \leq \rho_{ij} \leq 1 \quad (2.27)$$

The $p \times p$ matrix of population correlation coefficients is defined as

$$P = (\rho_{ij}) \quad (2.28)$$

The sample covariance matrix, S , and the sample

correlation matrix, R , may be found by replacing the population covariances and correlations with their maximum likelihood estimators. For example, the sample correlation coefficient between Y^i and Y^j is

$$r_{ij} = \frac{s_{ij}}{(s_{ii}s_{jj})^{1/2}} \quad -1 \leq r_{ij} \leq 1 \quad (2.29)$$

where s_{ij} is the maximum likelihood estimator of σ_{ij} .

We may wish to test hypotheses concerning correlation coefficients. Under the assumption of joint normality the monotonic transformation

$$z = \tanh^{-1} r_{ij} \quad (2.30)$$

produces an asymptotic normal variate with mean

$$q \approx 1/2 \log \left(\frac{1 + \rho_{ij}}{1 - \rho_{ij}} \right) \quad (2.31)$$

and variance

$$\text{Var}(z) = \frac{1}{N - 3} \quad (2.32)$$

as the number of observations, N , becomes large (15).

By using this z -transform we may test

$$H_0: \rho_{ij} = \rho_0$$

against

$$H_1: \rho_{ij} \neq \rho_0$$

We reject H_0 if

$$|z - q_0| \sqrt{N - 3} > Z_{\alpha/2} \quad (2.33)$$

where q_0 is the z -transform of $r = \hat{p}_0$ and $Z_{\alpha/2}$ is the upper 100(1 - α) percentage point of the standard normal distribution. Confidence intervals may be constructed concerning the ρ_{ij} and expressions for the power of the test are available (39).

Multiple Correlation

When considering a p -dimensional response vector, in addition to simple correlation, we will usually be interested in the multiple correlation of a given response component with the other $p-1$ response components, or a linear combination of the other $p-1$ response components. The multiple correlation coefficient, P_i , of one response component, Y^i , with a linear combination of the other $p-1$ response components is defined as

$$P_i = \max \text{corr}(Y^i, \underline{\alpha}' \underline{X}) \quad (2.34)$$

where $\underline{\alpha}$ is the $p-1$ dimensional contrast vector and \underline{X} is the vector of the other $p-1$ response variables. P_i is the largest possible correlation between Y^i and any linear combination of the other $p-1$ response variables (43). The quantity P_i^2 is called the population coefficient of multiple determination. The sample multiple correlation coefficient may be obtained from either the sample covariance matrix or the

sample correlation matrix. Let \tilde{S} be the sample covariance matrix computed from n independent observations. To find the multiple correlation coefficient for response component i , R_i , we rearrange the sample covariance matrix by replacing the 1st response with the i th response and partitioning the covariance matrix as follows

$$\begin{array}{cc} s_{11} & \tilde{S}'_{12} \\ \tilde{S}_{12} & \tilde{S}_{22} \end{array} \quad (2.35)$$

where s_{11} is now s_{ii} , \tilde{S}_{22} is the $p-1$ covariance matrix of the other $p-1$ remaining response components, and \tilde{S}_{12} is the $p-1$ vector of sample correlations between response i and the other $p-1$ response components. With the covariance matrix so partitioned

$$R_i^2 = R_1^2 = \frac{\tilde{S}_{12} \tilde{S}_{22}^{-1} \tilde{S}_{12}}{s_{11}} \quad (2.36)$$

We may be interested in testing the hypothesis

$$H_0: P_i = 0$$

against

$$H_1: P_i > 0$$

We would reject H_0 if

$$Q = \frac{R_i^2 (n-p)}{1-R_i^2 (p-1)} > F_{\alpha, p-1, n-p} \quad (2.37)$$

where n is the number of observations, p is the dimension of the response, and F is the upper $100(1 - \alpha)$ percentage point of the F distribution (43).

Independence of k Variates

We may be interested in determining whether a set of k multivariate normal response variates is independent. An appropriate hypothesis to test would be

$$H_0: \underline{P} = I$$

against

$$H_1: \underline{P} \neq I$$

where \underline{P} is the $k \times k$ population correlation matrix and I is the $k \times k$ identity matrix. We would reject H_0 if

$$\chi_0^2 = -(N - 1 - \frac{2k+5}{6}) \log |\underline{R}| > \chi_{\alpha, 1/2k(k-1)}^2 \quad (2.38)$$

where \underline{R} is the $k \times k$ sample correlation matrix, N is the number of independent observations from $N(\underline{\mu}, \underline{\Sigma})$, and χ^2 is the upper-tail χ^2 distribution (9,39). This test would be appropriate before proceeding with any multivariate analysis.

Independence of k Sets of Variates

In addition to determining whether a set of variates is independent, we will often be interested in determining whether k sets of multivariate normal variates are mutually independent. If the j th of k sets contains p_j variates, then we may partition the gross covariance matrix into submatrices $\underline{\Sigma}_{ij}$ of dimension $p_i \times p_j$. An appropriate hypothesis

to test would be

$$H_0: \Sigma_{ij} = \phi \quad \text{for } i \neq j$$

against

$$H_1: \Sigma_{ij} \neq \phi$$

For a sample of size N independent observations from a multivariate normal population, we would compute \underline{R} , the sample correlation matrix, and partition it as above. To test H_0 we use a test statistic due to Wilks

$$V = \frac{|R|}{|R_{11}| |R_{22}| \dots |R_{kk}|} \quad (2.39)$$

Box has shown that the statistic

$$x_0^2 = - \frac{(N-1)}{c} \log V \sim x_{\alpha, f}^2 \quad (2.40)$$

where

$$c^{-1} = 1 - \frac{(2S_1 + 3S_2)}{12f(N-1)} \quad (2.41)$$

$$f = S_2/2 \quad (2.42)$$

$$S_j = \left(\sum_{i=1}^k p_i \right)^j - \sum_{i=1}^k p_i^j \quad j=1,2 \quad (2.43)$$

We would reject H_0 if $x_0^2 > x_{\alpha, f}^2$ (11,39,57).

Generation of Multivariate Normal Random Vectors

Generation of Univariate Normal Random Variates

In order to investigate the MANOVA power function, we will require a procedure with which to generate multivariate normal random vectors. In order to generate these random vectors we require a procedure with which to generate independent univariate normal variates. Box and Muller have derived a direct transformation of uniform deviates which produces the desired variates. Let U_j and U_{j+1} be independent deviates from a Uniform (0,1) distribution; these deviates may be obtained from any valid uniform deviate generator. In order to generate variates from $N(\mu, \sigma^2)$ we transform the uniform deviates as follows:

$$x_j = \mu + (-2\sigma^2 \log U_j)^{1/2} \cos(2\pi U_{j+1}) \quad (2.44)$$

$$x_{j+1} = \mu + (-2\sigma^2 \log U_j)^{1/2} \sin(2\pi U_{j+1}) \quad (2.45)$$

x_j and x_{j+1} will be independent variates from $N(\mu, \sigma^2)$ (14).

Generation of Multivariate Normal Random Vectors

In order to generate p -dimensional random vectors from the multivariate normal population $N(\underline{\mu}, \underline{\Sigma})$ we use a fundamental theorem from multivariate analysis. If $\underline{z}' = (z_1, z_2, \dots, z_p)$ are p independent observations from $N(0, 1)$, then the p -dimensional vector, \underline{x} from $N(\underline{\mu}, \underline{\Sigma})$ may be represented as

$$\underline{X} = \underline{C}\underline{Z} + \underline{\mu} \quad (2.46)$$

where \underline{C} is a unique lower triangular matrix satisfying
(58)

$$\underline{\Sigma} = \underline{C}\underline{C}' \quad (2.47)$$

The matrix \underline{C} may be computed by a computational routine devised by Scheuer and Stoller (50).

Hence the generation of \underline{X} from $N(\underline{\mu}, \underline{\Sigma})$ takes three steps:

1. Computation of the \underline{C} matrix.
2. Generation of p independent variates from $N(0, 1)$.
3. Application of (2.46) above.

This procedure will be of great utility in studying the MANOVA power function.

CHAPTER III

MANOVA POWER GENERATION

Introduction

In order to make any meaningful analysis possible, we require a procedure which will enable us to obtain the power of the MANOVA test in a form useful to us in operational testing. Gnanadesikan has investigated the MANOVA power function in terms of the noncentrality parameters of the characteristic roots of a matrix using Monte Carlo methods (48). However, the noncentrality parameters, as well as the distribution of the characteristic roots of a matrix, are of minimal use in operational testing. We require the power of the MANOVA test in terms of the MANOVA model. We will propose a procedure with which we may determine the power of the MANOVA test in terms of departures from the model assumptions. This procedure should be of greater use to operational testers than the procedure involving the noncentrality parameters of the characteristic roots of a matrix.

MANOVA Power Criteria

Under the usual MANOVA assumptions we would be interested in determining the power of the test,

$$P\{\text{Reject } H_0 | H_0 \text{ is false}\}, \quad (3.1)$$

in terms of the hypothesis we are testing. In the ANOVA for a given test on a main effect, say factor A, we would be interested in the power of the test to detect departures from the assumptions of the form

$$\frac{\sum_{i=1}^a \alpha_i^2}{\sigma^2} = \text{constant} \quad (3.2)$$

It seems appropriate that a similar form may be of use for the MANOVA. We will now propose three useful forms of the MANOVA power criterion.

A typical hypothesis which we test in the MANOVA is that of no main effect. Consider the two-factor, p-dimensional MANOVA model. One hypothesis which we might test would be that of no main effect due to factor A,

$$H_0: \alpha_i = \phi \quad i=1, \dots, a$$

against

$$H_1: \alpha_i \neq \phi \quad \text{for at least one } i$$

It seems that we would now be interested in the power of the test to detect departures from the assumptions in norm of the form

$$D_2 = \left\| \frac{\sum_{i=1}^a (\alpha_i^1)^2}{\sigma_{11}}, \frac{\sum_{i=1}^a (\alpha_i^2)^2}{\sigma_{22}}, \dots, \frac{\sum_{i=1}^a (\alpha_i^p)^2}{\sigma_{pp}} \right\| \quad (3.3)$$

where D_2 is the euclidean norm of the p component departures.

We may also be interested in the power of the test to detect departures from the assumptions of the form

$$D_s = \max_j \frac{\sum_{i=1}^a (\alpha_i^j)^2}{\sigma_{jj}} \quad j=1, \dots, p \quad (3.4)$$

where D_s is the supremum norm of the p component departures. From the above it is clear that for the general p -dimensional case

$$D_s \leq D_2 \leq \sqrt{p} D_s \quad (3.5)$$

In addition to the criteria proposed above, we may be interested in detecting one of the p component departures at some level with the other $p-1$ components at some much lower random levels. For example, we might desire to detect

component j at $\frac{\sum_{i=1}^a (\alpha_i^j)^2}{\sigma_{jj}} = D_j$ and the other $p-1$ component departures at levels from the distribution Uniform $(0, D_j/R)$ where $R = 1, 2, \dots$, to be selected. This third criterion will be especially useful when comparing the relative effectiveness of ANOVA with MANOVA.

Monte Carlo Power Generation

Introduction

A Monte Carlo approach to determining the power of the MANOVA seems appropriate as the MANOVA power function

is not available in a usable form. This approach is especially inviting when we consider the wide availability in the U. S. Army of fast, large, time-shared computer systems. Our general approach will be to generate random observations which satisfy the MANOVA model of interest and the size and type component departures we desire to detect. Once we generate the observations, we compute the MANOVA test to determine whether to reject the null hypothesis and record the results. We repeat this procedure a large number of times. The power of the test will then be the ratio of the number of times we rejected the null hypothesis to the total number of tests we conducted.

In addition to the usual MANOVA calculations, with sample size, n , and component departures we desire to detect, we must be able to accomplish the following:

1. Randomly assign the p component departures in such a manner that they satisfy the MANOVA power criterion we desire to use.

2. For each $j=1, \dots, p$, randomly assign the a components, α_i^j , of each D_j such that $\frac{\sum_{i=1}^a (\alpha_i^j)^2}{\sigma_{jj}} = D_j$ and $\sum_{i=1}^a \alpha_i^j = 0$.

3. Obtain an estimate of the response correlation structure in the form of a $p \times p$ correlation matrix.

4. Randomly generate p -dimensional error vectors from $N(\underline{\mu}, \underline{\Sigma})$.

Due to the nature of our form of the MANOVA test procedure

it is not necessary to assign departure components to factors other than the one under consideration.

Random Assignment of Departure Components

In order to randomly assign the p departure components, we follow three different procedures depending upon whether we use the euclidean norm, the supremum norm, or set one component departure and randomly assign the other components small departures. For the euclidean norm we use the following procedure where r_j is a random variate with Uniform $(0,1)$ distribution.

1. $j = 0, Y = D_2^2$
2. $j = j + 1$
3. $D_j = Y \times r_j$
4. $Y = Y - D_j$
5. If $j < p-1$, go to step 2
6. $D_p = Y$
7. $D_j = (D_j)^{1/2} \quad j=1, \dots, p$

For the supremum norm we use the following procedure where r_j is a random variate with Uniform $(0,1)$ distribution.

1. $j = 0, Y = D_s$
2. $j = j + 1$
3. $D_j = Y \times r_j$
4. If $j < p-1$ go to step 2
5. $D_p = Y$

In order to insure randomness the order in which the p component departures are assigned would be randomized for both the

euclidean and supremum norms.

For the case in which we desire to set one component departure, D_k , to a certain level and the others to small random levels, we use the following procedure where r_j is a random variate with Uniform (0,1) distribution and $R=1,2,\dots$, to be selected.

1. $j = 0, Y = D_k$
2. $j = j + 1$
3. If $j = k, D_j = Y$, go to step 2
4. $D_j = r_j \times Y/R$
5. If $j < p$, go to step 2

In this case we would not randomly assign the order in which we assign the component departures for obvious reasons.

Random Assignment of Component Factor Levels

In randomly assigning the a factor levels for each component departure we face an additional constraint. We must randomly select a point from the set

$$\underline{\alpha} \in \left\{ \sum_{i=1}^a \alpha_i^2 = D_j \cap \sum_{i=1}^a \alpha_i = 0 \right\}.$$

This set is the intersection of the boundary of a closed ball with a hyperplane in euclidean a -space. For $a > 2$, the problem is not easily solved except by resorting to a non-linear optimization procedure:

$$\text{Minimize } \sum_{i=1}^a \alpha_i$$

$$\text{Subject to: } \sum_{i=1}^a \alpha_i^2 = D_j$$

by starting at a random starting point and using a penalty method. The problem may be solved best by a computer algorithm due to Bazaraa on the order of 10 to 100 milliseconds of CPU time (10).

If we are interested in saving computer time we may resort to an approximate method which yields fairly close results. The procedure is as follows where r_k is a random variate distributed Uniform (0,1).

1. $N = a/2$, $REM = a \bmod 2$
2. $k = 0$
3. $k = k + 1$
4. If $N = 1$, go to step 8
5. $D_k = r_k \times D_j$
6. $D_j = D_j - D_k$
7. If $k < N-1$, go to step 3
8. $D_{k+1} = D_j$
9. $k = 0$, $l = 0$
10. $k = k + 1$
11. $l = l + 2$
12. $\alpha_{\ell}^j = -D_k / \sqrt{2}$
13. $\alpha_{\ell+1}^j = D_k / \sqrt{2}$
14. If $k < N$, go to step 10
15. If $REM = 1$, $\alpha_{\ell+2}^j = 0$

Again in order to insure randomness we will randomly assign the order in which we assign the component factor levels. This procedure will indicate a slightly lower power for even numbers of levels and a slightly higher power for odd numbers of levels. See Appendix A for details.

Response Correlation Structure

In the MANOVA we consider the response correlation structure to be reflected by the error correlation or covariance matrix. Two situations will generally prevail concerning our knowledge of the correlation structure in operational testing. In the first situation we will have some objective estimate of the correlation structure in the form of a sample correlation or covariance matrix obtained from prior experimentation. In the second situation we will have no objective estimate of the correlation structure, but we will have some subjective estimate. This subjective estimate of the correlation structure, in the form of a correlation matrix, will usually be obtained from the project managers and combat developers of the systems under consideration. Rarely, if ever, will there be no knowledge of the response correlation structure.

Random Generation of Error Vectors

In order to generate multivariate normal p -dimensional random error vectors from $N(\mu, \Sigma)$, we apply the results reviewed in Chapter II. In order to do so we require a

generator of Uniform (0,1) random deviates and a generator of Normal (0,1) variates. We then follow the procedure previously outlined:

1. Compute the $p \times p$ matrix \underline{C} , such that $\underline{C} \underline{C}' = \underline{\Sigma}$
 2. Generate a $p \times 1$ vector \underline{Z} whose components are independent variates from $N(0,1)$
 3. Apply the transformation $\underline{X} = \underline{C}\underline{Z} + \underline{\mu}$. The vector \underline{X} will be from the population $N(\underline{\mu}, \underline{\Sigma})$.
- N applications of the above procedure will result in N mutually independent p -dimensional vectors from $N(\underline{\mu}, \underline{\Sigma})$.

MANOVA Power Generation Procedure

In order to simplify our computations we will use a standardization transformation on all the responses. We will apply the following transformation

$$y^{jj'} = \frac{y^j - \mu^j}{(\sigma_{jj})^{1/2}} \quad (3.6)$$

For original y distributed $N(\underline{\mu}, \underline{\Sigma})$, the transformed y' will be distributed $N(\underline{\phi}, \underline{P})$, where \underline{P} is the population correlation matrix. This transformation will greatly simplify the MANOVA power calculations and permit us to express the component departures in standardized units of component variances of 1. The transformation will not effect the MANOVA test statistic because of the well known result that the determinant of a covariance matrix is equal to the product of the

component variances multiplied by the determinant of the corresponding correlation matrix.

The procedure we will use to determine the power of the MANOVA test for a given probability of Type I error, α , sample size, n , and correlation matrix, \underline{P} , is as follows:

1. Select the MANOVA model, for example, a completely-crossed, two factor, p -dimensional MANOVA model.
2. Estimate the response correlation structure, \underline{P} .
3. Select the hypothesis we desire to test, for example, no effect due to factor A.
4. Select the type and size component departures we desire to detect.
5. Select the number of Monte Carlo iterations, NR , we desire to run.
6. For each Monte Carlo iteration, randomly assign the component departures and component departure levels, as appropriate.
7. For each model index combination, for example the two-factor MANOVA model above, i, j, k , generate an error vector \underline{e}_{ijk} from $N(\underline{0}, \underline{P})$ and apply the model with all effect levels zero except the effect being tested, for example,

$$y_{ijk} = \alpha_i + e_{ijk}.$$
8. Compute the MANOVA test statistic, compare it with the critical value of the test, and record the results.
9. Repeat steps 5 - 8 $NR - 1$ times.

10. Compute the power of the MANOVA test:

$$\text{Power} = \frac{\text{number of hypotheses rejected}}{\text{NR}}$$

In his Monte Carlo power studies Gnanadesikan found that NR = 500 is adequate; the author has found the same number to be adequate also.

A complete FORTRAN IV program with necessary subroutines for use on a UNIVAC 1108 computer system, written and validated by the author, appears at Appendix A. This program is written to be used to find the power of the MANOVA test for a main effect in a two-factor, p-dimensional MANOVA model for $p = 1, \dots, 20$; $\alpha = .00, \dots, 1.00$; and any applicable number of factor levels up to twenty for each factor. The program permits use of either one of the three power criteria. The program uses the approximate method of assigning the component departure levels. It is entirely interactive and requests any information it requires.

A formal validation of the power program was not accomplished; however, each subroutine, as well as the main program, was validated separately. In addition, the multivariate normal generator portion of the program was tested for goodness of fit of the marginal distributions using a ten-cell χ^2 goodness-of-fit test at $\alpha = .05$. Using sample sizes of 10,000 with different mean vectors and covariance matrices, the generator at no time produced a sample which caused rejection at the stated confidence level. The sample

covariance matrix entries computed from the different samples never varied from the true values by more than five percent.

Although an investigation of the MANOVA power function was beyond the scope of this research, the author investigated a number of different two-factor MANOVA models. The following general statements concerning the power of the two-factor MANOVA tests were found to hold for a wide variety of problems:

1. Power is a decreasing function of the dimension of the response.
2. Power is an increasing function of the size departure to detect.
3. Power is an increasing function of sample size.
4. Power is an increasing function of the probability of Type I error, α .
5. Power is an increasing function of $-\log|P|$, where P is the correlation matrix of the multiresponse.

CHAPTER IV

DEVELOPMENT OF THE METHODOLOGY

Introduction

We return now to the primary objective of this research: to develop a methodology for use in comparing the applicability and effectiveness of ANOVA with MANOVA for use in the operational test and evaluation of alternative command and control systems. Clearly, MANOVA is the preferred procedure for use in evaluating systems with correlated measures of effectiveness as it provides for joint comparison of the measures. On the other hand, we may always conduct ANOVA on each individual measure at the cost of forsaking the systems approach to comparison. We initially proposed to compare the applicability and effectiveness of ANOVA with MANOVA using the following four factors:

1. The assumptions required for each model.
2. The effects of departures from the required assumptions.
3. The powers of the tests versus correlation, sample size, and the probability of type I error.
4. The validity of probability statements concerning systems parameters.

During our survey of the literature we discovered that little

research has been conducted concerning the effects of departures from MANOVA assumptions. On the other hand, a fair amount of research has been conducted concerning the effects of departures from ANOVA assumptions. Due to the great disparity between the known results in the two areas, we are unable to make valid comparisons on the basis of factors 1 and 2. However, it is clear that when departures from assumptions are suspected, ANOVA is preferred to MANOVA as the effects of departures are fairly well known for ANOVA.

As stated previously, we clearly prefer MANOVA to ANOVA for use on systems with correlated measures of effectiveness. Using MANOVA techniques, we may construct joint confidence intervals on systems parameters. We may not do so when utilizing ANOVA on each measure of effectiveness individually. Obviously, we may not compare ANOVA with MANOVA on the basis of factor 4, other than to note that MANOVA provides joint probability statements, whereas ANOVA does not.

We may, however, compare ANOVA with MANOVA on the basis of factor 3. We have developed in Chapter III a procedure with which we may determine the power of the MANOVA in a form useful in comparison of ANOVA with MANOVA. We will concentrate our efforts now on developing a methodology for comparing the effectiveness of ANOVA with MANOVA on the basis of factor 3. We will assume hereafter that the system

in question meets the assumptions for both the ANOVA and MANOVA models.

Segregating the Measures of Effectiveness

Separation of Independent Measures

Clearly, a comparison of the effectiveness of ANOVA with MANOVA is not applicable for independent measures of effectiveness. Given the overall set of measures of effectiveness, our first task should be to separate all independent measures from the rest. We accomplish this task by examining the measures of effectiveness correlation matrix. If our estimate of the correlation matrix is subjective, we separate those measures which have zero simple correlation with all other measures. These independent measures will form the set, I , of mutually independent measures.

If our estimate of the correlation matrix is objective, we may use certain statistical tests to help us separate the independent measures. Consider that we have p systems measures of effectiveness. We would compute the sample multiple correlation coefficients, R_i , $i = 1, \dots, p$, using the procedure of 2.36. Then we would test each of the p hypotheses of the form

$$H_0: P_i = 0$$

against

$$H_1: P_i > 0$$

using the procedure of (2.37). Those measures for which we fail to reject the above hypotheses will form the set, I , of mutually independent measures. Whether our estimate of the correlation structure is objective or subjective, we will obtain a set of mutually independent measures; the set may be empty. These measures must be analyzed using ANOVA.

Grouping of Independent Sets of Measures

After separating the independent measures from the rest, we would like to group the remaining measures into k sets of measures which are correlated within sets, but independent between sets. Let us designate these sets C_i , $i=1, \dots, k$. If our estimate of the correlation structure is subjective, we make the groupings on the basis of the subjective correlation matrix only. If our estimate of the correlation structure is objective, we may, in addition to grouping based upon the correlation matrix, test whether the sets of measures we have formed are mutually independent using the procedure of (2.39) and (2.40). Also, we may test to insure that each set is correlated using the procedure of (2.38).

For these k independent sets of correlated measures, C_i , $i=1, \dots, k$ MANOVA is the appropriate procedure to utilize. With the completion of grouping each measure of effectiveness into the set of independent measures, I , or one of the sets of correlated measures, C_i , $i=1, \dots, k$, we are ready to proceed with the comparison of the effectiveness of ANOVA with

MANOVA for each correlated set, C_i , $i=1, \dots, k$.

Determining the Powers of the Tests

ANOVA Power

For each measure of effectiveness, the operational testor must specify:

1. α , the probability of Type I error desired.
2. n_{\max} , the maximum sample size permitted.
3. $\sum_{i=1}^a \alpha_i^2 / \sigma^2 = D$, the component departure to detect.
4. $(1 - \beta)$, the power of the test desired.

α and $(1 - \beta)$ should be the same for each measure; however, n_{\max} and D could vary. Given the above we may determine the following:

1. The sample size that is required for the desired power, n_{anova} .
2. The power corresponding to the maximum sample size permitted, n_{\max} .

1 and 2 would be accomplished using the procedures outlined in Chapter II.

In some cases the maximum size permitted, n_{\max} , may be insufficient to achieve the desired power. When this situation arises, it must be reconciled by adjusting either D , or n_{\max} , or both. In any case, the ANOVA sample size which is required for the desired power, n_{anova} , must be computed for each measure of effectiveness.

MANOVA POWER

In addition to the parameters provided for each individual measure of effectiveness, the operational testor must specify for each independent set of correlated measures, C_i , $i=1, \dots, k$, the following:

1. α , the joint probability of Type I error.
2. $(1 - \beta)$, the joint power desired.
3. R , the ratio of the primary component departure to the maximum departure of the other components.

The maximum sample sizes, permitted, as well as the departures to detect, would be previously specified. The joint α and $(1 - \beta)$ should be the same as those specified for the ANOVA.

For each correlated set, C_i , $i=1, \dots, k$, we will have p_i measures of effectiveness. For each measure of effectiveness Y^j , $j=1, \dots, p_i$, we will have its associated α , $(1-\beta)$, $n_{\max j}$, $n_{\text{anova } j}$ and D_j . For each set C_i , $i=1, \dots, k$, we will have a joint α , $(1-\beta)$, and R_i . As stated previously the α , and $(1 - \beta)$ should be the same for both joint and univariate cases.

Given the parameters for each correlated set, C_i , $i=1, \dots, k$, we are interested in determining the power of the MANOVA in such a way as to permit us to compare its effectiveness with that of the ANOVA. Ideally, we would want the MANOVA test to reject the null hypothesis if only one of the p_i measure departures is not zero. One approach would

be, for each measure in the set, Y^j , $j=1, \dots, p_i$, in turn, set its component departure to its corresponding D_j and all other component departures to zero and determine the sample size required to achieve the desired power, $n_{\text{manova } j}$. Should this sample size be less than or equal to the sample size required to achieve the same ANOVA power, $n_{\text{anova } j}$, then we would consider MANOVA more effective than ANOVA for that measure.

Perhaps, the arbitrary setting of the other p_i-1 measure departures to zero may be unrealistic. It would be more intuitively appealing to set the other p_i-1 measure departures to some random level significantly lower than the primary measure departure. Let us randomly assign these other p_i-1 measure departures levels which are distributed. Uniform $(0, D_j/R_i)$ where D_j is the desired departure to detect for the primary measure under consideration and R_i is the ratio provided by the operational testor.

For each measure in a correlated set, Y^j , $j=1, \dots, p_i$ we would apply the above procedure, finding the sample size required for the desired MANOVA power, $n_{\text{manova } j}$. After completing the above procedure we would have for each measure in the correlated set:

1. α , the probability of Type I error.
2. $(1 - \beta)$, the power desired.
3. D_j , the measure departure to detect.
4. $n_{\text{max } j}$, the maximum sample size permitted.

5. $n_{\text{anova } j}$, the ANOVA sample size required to achieve the desired ANOVA power.

6. $n_{\text{manova } j}$, the MANOVA sample size required to achieve the desired MANOVA power.

Trading Joint Inference for Power

For a correlated set of measures, C_i , $i=1, \dots, k$, we are constrained by the minimum sample size in the set, $n_{\min} = \min_j(n_{\text{anova } j})$, so far as MANOVA sample size is concerned for the system as a whole. If we are unable to achieve the desired MANOVA power for each of the p_i measures in the set using the n_{\min} , then we must develop a procedure to remove measures from the correlated set, one at a time, so as to improve the MANOVA power of the remaining $p_i - 1$ measures in the set. We already know that the MANOVA power will improve monotonically with a decrease in the dimension of the response. Since we are constrained by n_{\min} , it would seem that the appropriate measure to remove would be the measure corresponding to the n_{\min} . In the event of more than one measure corresponding to the n_{\min} , it would seem reasonable to remove the measure with the smallest MANOVA power. After removing the measure selected we would recompute the $n_{\text{manova } j}$ for the remaining measures and the n_{\min} of the remaining measures. If at this time all of the measures remaining in the set achieve the desired power with $n_{\text{manova } j}$ less than or equal to n_{\min} , we stop; MANOVA is more effective

than ANOVA for those measures remaining in the set. If we are not able to achieve the desired power with all $n_{\text{manova } j}$ less than or equal to n_{min} , we remove the appropriate measure and repeat the procedure.

In the course of carrying out the above analysis, one of two situations will be encountered. Either we will terminate the procedure with more than one measure remaining in the set, or we will terminate the procedure with one measure remaining in the set. In the former case MANOVA is more effective than ANOVA for the measures remaining in the set, and ANOVA is more effective than MANOVA for the measures removed from the set. In the latter case ANOVA is more effective than MANOVA for all measures.

Summary of the Methodology

A summary of the methodology for comparing the effectiveness of ANOVA with MANOVA under the assumption that the system in question meets the required assumptions for each model is as follows:

1. Determine the correlation matrix for the measures of effectiveness.
2. Separate the measures of effectiveness into mutually independent sets of independent measures, I , and correlated measures, C_i , $i=1, \dots, k$.
3. Determine the probability of Type I error, α , and the power of the test, $(1 - \beta)$, to be utilized.

4. For each measure of effectiveness, determine the maximum sample size permitted, n_{\max} , and the univariate departure to be detected, D .

5. For each measure of effectiveness, determine the sample size, n_{anova} , required to achieve the required power. If $n_{\text{anova}} > n_{\max}$, reconcile the difference by adjusting D and/or n_{\max} .

6. For each set of correlated measures of effectiveness, C_i , $i=1, \dots, k$, perform the following.

a. For each measure of effectiveness, y^j , $j=1, \dots, p_i$, determine the sample size, $n_{\text{manova } j}$, required to achieve the desired MANOVA power with the measure under consideration departure set at D_j , and all remaining measure departures selected from Uniform $(0, D_j/R_i)$ where R_i is the ratio chosen by the testor.

b. If the $n_{\text{manova } j}$ are less than or equal to the $n_{\min} = \min_j(n_{\text{anova } j})$ for the desired power, stop; MANOVA is more effective than ANOVA for the measures in the set.

c. If the $n_{\text{manova } j}$ are greater than the n_{\min} for one or more measures in the set, remove from the set the measure corresponding to the n_{\min} . If more than one measure corresponds to the n_{\min} , remove from the set the measure with the lowest power which corresponds with the n_{\min} . Renumber all measures in the set which remain; set $p_i = p_i - 1$. If $p_i = 1$, stop; ANOVA is more effective than MANOVA for all original measures in the set C_i . If $p_i > 1$, repeat steps a through c.

In steps 6a through 6c above, we will achieve identical results by determining the MANOVA power for each measure of effectiveness using the n_{\min} at each stage and comparing the power achieved to the power required instead of determining the MANOVA sample size, n_{manova} , required to achieve the required power.

In the next chapter we will demonstrate the use of the methodology by working an example problem.

CHAPTER V

DEMONSTRATION OF THE METHODOLOGY

Introduction

In this chapter we will demonstrate the methodology which we developed in Chapter IV. We will use a hypothetical command and control system as vehicle. This hypothetical system will be known as the Brigade Antiarmor Command and Control System (BACCS). BACCS is designed for employment by U. S. Army airborne and airmobile brigades, units which are historically weak in antiarmor capabilities. Two competing forms of BACCS are under consideration for acquisition; these two forms are designated BACCS-I and BACCS-II. The composition of the two systems is similar; however, the two systems utilize different hardware.

For OT-II, the Commander, U. S. Army Operational Test and Evaluation Agency (OTEA), has approved a comparative operational test of the two systems consisting of three scenarios. The three scenarios are:

1. Brigade in the area defense.
2. Brigade in the defense of an airhead.
3. Brigade in the delay.

The Commander of OTEA has approved seven measures of effectiveness designated MOE-1 through MOE-7. In addition

the Commander of OTEA has approved a completely crossed two-factor experiment with equal numbers of observations per cell. He now desires to determine for which MOE MANOVA will be more effective, powerwise, than ANOVA, assuming that BACCS meets the assumptions required for both models.

Correlation Structure of the MOE

In this case we are fortunate to have an objective estimate of the correlation structure of the MOE. MANOVA was utilized for all seven MOE during OT-I; consequently, we have an estimate based upon an equivalent 42 observations. Recall that $\hat{\Sigma} = \frac{1}{ab(n-1)} E$ (2.23). The MOE correlation matrix is

	1	2	3	4	5	6	7
1	1.00	.00	-.06	-.12	.00	-.17	.16
2	.00	1.00	.01	-.11	.01	-.04	.76
3	-.06	.01	1.00	.68	-.49	.56	.07
4	-.12	-.11	.68	1.00	-.21	.72	-.04
5	.00	.01	-.49	-.21	1.00	-.26	-.11
6	-.17	-.04	.56	.72	-.26	1.00	-.08
7	.16	.76	.07	-.04	-.11	-.08	1.00

Were an objective estimate of the correlation structure not available, we would have to utilize a subjective estimate developed jointly by the BACCS project manager and the U. S. Army Training and Doctrine Command.

Based upon a knowledge of BACCS, we feel that only MOE-1 is independent of all other MOE. We desire to test the hypothesis

$$H_0: P_1 = 0$$

against

$$H_1: P_1 > 0$$

Using a computer program developed by the author (Appendix B) we compute the sample multiple correlation coefficient

$$R_1 = 0.293896$$

and

$$R_1^2 = 0.086375$$

Applying (2.37) we obtain the test statistic

$$Q = \frac{R_1^2(n-p)}{1-R_1^2(p-1)} = \frac{(.086375)(42-7)}{(1-.086375)(7-1)} = 0.5515.$$

We desire to test the hypothesis at $\alpha = .05$; therefore, the critical value of the test is $F_{.05,6,35} = 2.36$. The test statistic is less than the critical value of the test; hence, we fail to reject the hypothesis that MOE-1 is independent of the other MOE. We assign MOE-1 to the set of mutually independent measures, I.

Based upon our knowledge of BACCS, we feel that MOE-2 and MOE-7 are correlated, but independent of the other MOE.

We also feel that MOE-3, MOE-4, MOE-5, and MOE-6 are correlated, but are independent of the other MOE. We assign MOE-2 and MOE-7 to correlated set C_1 . We assign MOE-3, MOE-4, MOE-5, and MOE-6 to correlated set C_2 . The correlation matrix for the set C_1 is now the 2×2 matrix

$$\begin{array}{cc} 2 & 1.00 & .76 \\ 7 & .76 & 1.00 \end{array},$$

and the correlation matrix for the set C_2 is now the 4×4 matrix

$$\begin{array}{ccccc} 3 & 1.00 & .68 & -.49 & .56 \\ 4 & .68 & 1.00 & -.21 & .72 \\ 5 & -.49 & -.21 & 1.00 & -.26 \\ 6 & .56 & .72 & -.26 & 1.00 \end{array}$$

We desire to test the hypothesis that sets C_1 and C_2 are mutually independent using the procedure of (2.39) and (2.40) with $\alpha = .05$. Using a computer program developed by the author (Appendix C), we determine that the test statistic

$$\chi_0^2 = 3.5587$$

and the critical value of the test

$$\chi_{.05,8}^2 = 15.5073.$$

The test statistic is less than the critical value to the test; hence, we fail to reject the hypothesis of independence. We conclude that C_1 and C_2 are independent. One final check upon the MOE correlation structure we will make is to test the hypotheses

$$H_{10}: \rho_{C_1} = 1$$

against

$$H_{11}: \rho_{C_1} \neq 1$$

and

$$H_{20}: \rho_{C_2} = 1$$

against

$$H_{21}: \rho_{C_2} \neq 1.$$

We make this check to insure that the sets are, in fact, correlated.

Set C_1 contains only two members, MOE-2 and MOE-7; therefore, we may test another form of hypothesis H_{10} :

$$H_{10A}: \rho_{27} = 0$$

against

$$H_{11A}: \rho_{27} \neq 0.$$

We test hypothesis H_{10A} using the results of (2.30) through (2.33). The z-transformation of $r_{27} = .76$ is $z = \tan^{-1}(.76) = 0.638$. The test statistic is

$$|z| \sqrt{N-3} = 0.638 \sqrt{42-3} = 3.984.$$

The critical value of the test with $\alpha = .05$ is $Z_{.025} = 1.96$. The test statistic exceeds the critical value of the test; hence, we reject H_{10A} and conclude that MOE-2 and MOE-7 are correlated.

We test H_{20} using the results of (2.38) by means of a computer program developed by the author (Appendix D). The test statistic is

$$\begin{aligned} x_0^2 &= -(N - 1 - \frac{2k+5}{6}) \log |R_2| = -(42 - 1 - \frac{2 \cdot 4 + 5}{6}) \log |R_2| \\ x_0^2 &= 62.03133 . \end{aligned}$$

The critical value of the test is

$$\chi^2_{.05,6} = 12.59159 ,$$

with $\alpha = .05$. The test statistic exceeds the critical value of the test; hence, we conclude that the members of set C_2 are correlated.

With the conclusion of the above test for independence, we have completed the grouping of our MOE into three mutually independent sets:

$$I = \{\text{MOE-1}\}$$

$$C_1 = \{\text{MOE-2, MOE-7}\}$$

$$C_2 = \{\text{MOE-3, MOE-4, MOE-5, MOE-6}\} .$$

ANOVA is appropriate for MOE-1, the sole member of set I;

therefore, MOE-1 will not be used for the comparison of the effectiveness of MANOVA with ANOVA in later sections.

ANOVA Power/Sample Size for the MOE

The Commander of OTEA has specified that the following parameters be used for BACCS OT-II:

Probability of Type I error, - .05

Power of the test $(1 - \beta)$ - .75.

These parameters will apply to both ANOVA and MANOVA. Upon recommendation of the staff statisticians, the Commander of OTEA has designated, for each MOE, the maximum sample size permitted, n_{\max} , and the departure to be detected,

$$D = \frac{\sum_{i=1}^2 \alpha_i^2}{\sigma^2} . \quad \text{These parameters are shown in Table 1.}$$

Table 1. MOE Maximum Sample Sizes and Departures

MOE	Maximum Sample Size n_{\max}	Departure To Detect D
1	6	1.5
2	6	1.5
3	4	2.0
4	6	1.5
5	6	1.5
6	7	1.0
7	6	1.5

Using the above information, we compute, for each MOE, the minimum sample size, n_{anova} , required to achieve the power desired. We accomplish this by using the procedure reviewed in Chapter II. We use the Pearson-Hartley Charts in Hines and Montgomery (33). We enter the charts with the following parameters:

$$\alpha = .05$$

$$v_1 = a - 1 = 1$$

$$v_2 = ab(n-1) = 6(n-1)$$

$$\phi^2 = \frac{nD}{a} = \frac{nD}{2}.$$

The minimum sample sizes required for the ANOVA power desired, .75, and the probability of Type I error, $\alpha = .05$, are shown in Table 2.

Table 2. MOE Sample Sizes for Required Power

MOE	Maximum Sample Size n_{max}	Departures To Detect D	Minimum Sample Size n_{anova}
1	6	1.5	5
2	6	1.5	5
3	4	2.0	4
4	6	1.5	5
5	6	1.5	5
6	7	1.0	7
7	6	1.5	5

Comparing the Effectiveness of MANOVA with ANOVA

For the two correlated sets, C_1 and C_2 , we are now interested in determining for which members of these sets MANOVA is more effective than ANOVA from the standpoint of power. The Commander of OTEA has approved a ratio of $R = 2$ for use in setting the random levels of the MOE in the sets other than the one under consideration; therefore, the other MOE in the sets will have their departure levels assigned from the distribution Uniform $(0, D_j/2)$, where D_j is the departure of the MOE under consideration.

For set $C_1 = \{\text{MOE-2, MOE-7}\}$ we find that $n_{\min} = \min(n_{\text{anova } 2}, n_{\text{anova } 7}) = 5$ (Table 2). Using the two-factor MANOVA power program (Appendix I), we set $\alpha = .05$, levels of factor A = 2, levels of factor B = 3, $D = 1.5$, sample size = $n_{\min} = 5$, $R = 2$, Monte Carlo iterations = 500, and correlation matrix = P_{C_1} . The program yields the result: MANOVA power = .762. The MANOVA power is greater than the ANOVA power with sample size n_{\min} ; consequently, MANOVA is more effective than ANOVA for the members of set C_1 . We are able to achieve joint inference on both members of the set with the same sample size we would require to achieve inference on both members of the set with the same sample size we would require to achieve inference on the two MOE individually.

For set $C_2 = \{\text{MOE-3, MOE-4, MOE-5, MOE-6}\}$ we again

use the two-factor MANOVA power program. We set $\alpha = .05$, levels of factor A = 2, levels of factor B = 3, Monte Carlo iterations = 500, R = 2. For the four MOE in the set we determine $n_{\min} = 4 = n_{\text{anova } 3}$. We run the power program for each MOE with sample size = $n_{\min} = 4$ and size departure to detect, $D = D_j$, $j = 3, 4, 5, 6$. The results of these runs are shown in Table 3.

Table 3. MOE MANOVA Power I

MOE	MANOVA Sample Size n_{manova}	Departure To Detect D	Power
3	4	2.0	.614
4	4	1.5	.482
5	4	1.5	.496
6	4	1.0	.452

The n_{manova} required to attain the desired power, .75, is greater than the $n_{\min} = 4$ for one or more members of the set; therefore, we remove from the set the MOE which corresponds to the n_{\min} (MOE-3). We now determine the new n_{\min} . Set C_2 now contains MOE-4, MOE-5, and MOE-6. n_{\min} is now $n_{\text{anova } 4} = n_{\text{anova } 5} = 5$. With this new n_{\min} and the same parameters we used previously, we run the MANOVA power program for each of the remaining MOE with the results shown in Table 4.

Table 4. MOE MANOVA Power II

MOE	Manova Sample Size n_{manova}	Departure To Detect D	Power
4	5	1.5	.686
5	5	1.5	.646
6	5	1.0	.632

The n_{manova} required to attain the desired power, .75, is greater than the $n_{\text{min}} = 5$ for one or more members of the set; therefore, we remove the MOE which corresponds to the n_{min} . In this case both MOE-4 and MOE-5 correspond to the n_{min} ; consequently, we remove MOE-5, the MOE with the lower power for the same sample size. We now determine the new n_{min} . Set C_2 now contains only MOE-4 and MOE-6. n_{min} is now $n_{\text{anova } 4} = 5$. With this new n_{min} and the same parameters we used previously, we run the MANOVA power program for each of the two remaining MOE with the results shown in Table 5.

Table 5. MOE MANOVA Power III

MOE	Manova Sample Size n_{manova}	Departure To Detect D	Power
4	5	1.5	.782
6	5	1.0	.758

Both MOE in set C_2 now achieve the desired power with the n_{\min} . We conclude that for MOE-4 and MOE-6 MANOVA is more effective, powerwise, than ANOVA. In summary, we have found that ANOVA is more effective than MANOVA for MOE-3 and MOE-5; however, MANOVA is more effective than ANOVA for the sets $C_1 = \{\text{MOE-2, MOE-7}\}$ and $C_2 = \{\text{MOE-4, MOE-6}\}$. This information would be transmitted to the Commander of OTEA for his use in BACCS OT-II.

Although the example presented in this chapter is hypothetical, the same methodology can be applied to any system so long as an estimate of the correlation structure of the measures of effectiveness is available.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Limitations of the Research

This research has been limited by the initial assumptions of two-factor, fixed-effects, crossed models, equal sample sizes per cell, and no effect due to operators. In addition, due to the dearth of research concerning the effects of departures from MANOVA assumptions, the author limited the comparison of ANOVA with MANOVA to the consideration of the powers of the tests under the assumptions that the systems in question meet the assumptions required for both the ANOVA and MANOVA models and that an estimate of the correlation structure of the measures of effectiveness is available.

Conclusions

It is possible to formulate the MANOVA power criterion in a form which is meaningful to operational testors. One form of the MANOVA power criterion is the euclidean norm of the p individual response departures,

$$D_2 = \left\| \frac{\sum_{i=1}^a (\alpha_i^1)^2}{\sigma_{11}}, \frac{\sum_{i=1}^a (\alpha_i^2)^2}{\sigma_{22}}, \dots, \frac{\sum_{i=1}^a (\alpha_i^p)^2}{\sigma_{pp}} \right\|$$

A second form of the MANOVA power criterion is the supremum norm of the p individual response departures,

$$D_s = \max_j \frac{\sum_{i=1}^a (\alpha_i^j)^2}{\sigma_{jj}}$$

A third form of the MANOVA power criterion, one useful in comparing the power of the ANOVA with that of the MANOVA, is for $k=1, \dots, p$, in turn

$$D_k = \frac{\sum_{i=1}^a (\alpha_i^k)^2}{\sigma_{kk}} \quad \text{and} \quad D_j \sim \text{Uniform}(0, D_k/R)$$

for $j=1, \dots, p$; $j \neq k$.

The methodology developed in this research, utilizing the third form of the MANOVA power criterion, is a valid method by which to compare the effectiveness of ANOVA with MANOVA under the assumptions already mentioned.

Recommendations

Several recommendations for future research arose in the course of this research. One recommendation is to investigate the effect of serial correlation on the power of the MANOVA test using the MANOVA power program developed by the author. Another recommendation is to extend the MANOVA power program developed by the author so that it may handle nested,

multi-factor designs. The final recommendation is that the U. S. Army Operational Test and Evaluation Agency implement the methodology developed in this research for use in its operational testing program.

A P P E N D I C E S

APPENDIX A

This appendix contains a complete FORTRAN listing of the two-factor MANOVA power program along with an example of its use. The main program controls the subroutines, computes the MANOVA test statistics, and records the results of the tests. The program is entirely interactive; input is made in free-field format. The program listing follows the example of its use.

@XQT TMANOVA.POWER

** MANOVA POWER PROGRAM **

ENTER THE NR OF STARTUP RUNS FOR UNIF
753

ENTER THE NR LEVELS OF FACTOR A
2

ENTER THE NR LEVELS OF FACTOR B
3

ENTER THE DIMENSION OF THE RESPONSE
4

ENTER THE SAMPLE SIZE
4

ENTER ALPHA
.05

DO YOU DESIRE TO SPECIFY ALL NORM COMPONENTS?
YES

ENTER THE NORM INDEX TO BE SPECIFIED
1

WHAT NORM RATIO DO YOU WANT TO USE?
2.

ENTER THE SIZE NORM YOU DESIRE TO DETECT
2.

ENTER THE ITERATIONS SAMPLE SIZE
500

ENTER THE SIGMA MATRIX
1.,.68,-.49,.56
.68,1.,-.21,.72
-.49,-.21,1.,-.26
.56,.72,-.26,1.

ENTER THE MEAN VECTOR
0.,0.,0.,0.

** STARTUP RUNS FOR UNIF = 753

** LEVELS OF FACTOR A = 2

** LEVELS OF FACTOR B = 3

```

** SAMPLE SIZE = 4
** VECTOR DIMENSION = 4
** ITERATIONS SAMPLE SIZE = 500
** ALPHA = .05
** SIZE NORM TO DETECT = 2.00
** NORM 1 IS SPECIFIED
** NORM RATIO IS 2.00

```

```

** SIGMA MATRIX **

```

1.0000	.6800	-.4900	.5600
.6800	1.0000	-.2100	.7200
-.4900	-.2100	1.0000	-.2600
.5600	.7200	-.2600	1.0000

```

** C MATRIX **

```

1.0000	.0000	.0000	.0000
.6800	.7351	.0000	.0000
-.4900	.1671	.8567	.0000
.5600	.4589	-.0665	.6842

```

** MEAN VECTOR **

```

.0000	.0000	.0000	.0000
-------	-------	-------	-------

IS YOUR INPUT CORRECT ?
YES

```

** POWER OF THE TEST **

```

SAMPLE SIZE	POWER OF THE TEST
4	.61400

DO YOU DESIRE TO MAKE ANOTHER RUN ?
NO

NORMAL EXIT. EXECUTION TIME 48812 MLSEC.

```

INTEGER ERROR
COMMON /ONE/ E(20,20),H1(20,20),H0(20,20)
COMMON /TWO/ NL
COMMON /THREE/ KORD(20)
COMMON /FOUR/ KUSED(20)
COMMON /FIVE/ SIGMA(20,20)
COMMON /SIX/ CMAT(20,20)
COMMON /SEVEN/ ZVEC(20),U(20),XVEC(20),BUF(20)
COMMON /NINE/ FAC(20)
COMMON /ELEVEN/ NI
DIMENSION DCOM(20)
DIMENSION A(3,20),Y(3,5,100,20),C(3,5,20),T(5,20),R(3,20)
DIMENSION G(20),JD(20),V(2),H2(20,20),H3(20,20),Z(20,20)
DIMENSION IEUC(2),ISUP(2)
DATA IEUC/6HEUCLID,6HEAN /
DATA ISUP/6HSUPREM,6HUM /
DATA KNORM/6HEUC /
0001 FORMAT( )
0101 FORMAT(1H1,2X,'ERROR**READ PAST END OF FILE**')
0103 FORMAT(1H1,2X,'ERROR**PROBLEM IN CHI SQUARED ROUTINE**')
0105 FORMAT(1H1,2X,'ERROR**PROBLEM IN GUR ROUTINE**')
0112 FORMAT(1H1,10X,'** MANOVA POWER PROGRAM **')
0114 FORMAT( /,10X,'** ALPHA =',F5.2)
0116 FORMAT( /,10X,'** VECTOR DIMENSION = ',I2)
0118 FORMAT( /,10X,'** POWER OF THE TEST **')
0120 FORMAT( /,10X,'** SIZE NORM TO DETECT = ',F5.2)
0121 FORMAT( /,10X,'** LEVELS OF FACTOR A = ',I3)
0122 FORMAT( /,10X,'SAMPLE SIZE',9X,'POWER OF TEST')
0123 FORMAT( /,10X,'** LEVELS OF FACTOR B = ',I3)
0124 FORMAT(/,14X,I3,16X,F10.5)
0125 FORMAT( /,10X,'** SAMPLE SIZE = ',I3)
0127 FORMAT( /,10X,'** ITERATIONS SAMPLE SIZE = ',I3)
0128 FORMAT( /,10X,'** MEAN VECTOR **')
0131 FORMAT( /,10X,'** SIGMA MATRIX **')
0133 FORMAT( /,10X,'** C MATRIX **')
0137 FORMAT( /,2X,'ENTER THE SIGMA MATRIX')
0139 FORMAT( /,2X,'ENTER THE MEAN VECTOR')
0141 FORMAT( /,2X,8(1X,F8.4))
0143 FORMAT( /,2X,'IS YOUR INPUT CORRECT ?')
0144 FORMAT( /,2X,'ENTER THE TYPE NORM YOU DESIRE TO USE: EITHER')
0145 FORMAT(,2X,'EUC FOR EUCLIDEAN OR SUP FOR SUPREMUM')
0146 FORMAT( /,10X,'** NORM USED IS ',2A6)
0147 FORMAT( //,2X,'** DO YOU DESIRE TO MAKE ANOTHER RUN ?')
0148 FORMAT( /,2X,'DO YOU DESIRE TO CHANGE ONLY SAMPLE SIZE, ALPHA,
1 NORM ?')
0149 FORMAT( /,2X,'ENTER THE SAMPLE SIZE')
0150 FORMAT( /,2X,'ENTER ALPHA')
0151 FORMAT(A6)
0152 FORMAT( /,2X,'ENTER SIZE NORM YOU DESIRE TO DETECT')
0153 FORMAT( /,2X,'ENTER THE NR OF LEVELS OF FACTOR A')
0154 FORMAT( /,2X,'ENTER THE NR OF LEVELS OF FACTOR B')
0155 FORMAT( /,2X,'ENTER THE DIMENSION OF THE RESPONSE')
0157 FORMAT( /,2X,'ENTER THE ITERATIONS SAMPLE SIZE')
0158 FORMAT( /,2X,'ENTER THE NR OF STARTUP RUNS FOR UNIF')
0159 FORMAT( /,10X,'** STARTUP RUNS FOR UNIF = ',I5)
0160 FORMAT(/,2X,'DO YOU DESIRE TO SPECIFY ALL NORM COMPONENTS?')

```

```

0161 FORMAT(/,2X,'ENTER THE NORM INDEX TO BE SPECIFIED')
0162 FORMAT(/,10X,'** DCOM('',12,'') =',F5.2)
0163 FORMAT(/,10X,'** NORM '',12,' IS SPECIFIED')
0164 FORMAT(/,2X,'WHAT NORM RATIO DO YOU WANT TO USE?')
0165 FORMAT(/,10X,'** NORM RATIO IS ',F5.2)
      DATA IRES/6HYES /
      EXTERNAL UNIF,RNORM1,CMAT1,CHI,CHIN

```

C

C ** INPUT SECTION **

C

```

      WRITE(6,0112)
      WRITE(6,0158)
      READ(5,0001,END=9791) KSU
      DO 0800 I=1,KSU
        ZSU=UNIF(A)
0800 CONTINUE
0900 WRITE(6,0153)
      READ(5,0001,END=9791) NI
      WRITE(6,0154)
      READ(5,0001,END=9791) NJ
      WRITE(6,0155)
      READ(5,0001,END=9791) NL
      WRITE(6,0149)
      READ(5,0001,END=9791) NI4
      WRITE(6,0150)
      READ(5,0001,END=9791) ALPHA
      WRITE(6,0160)
      READ(5,0151) LNOR
      IF(LNOR.NE.IRES) GO TO 0902
      WRITE(6,0161)
      READ(5,001) IDX
      WRITE(6,0164)
      READ(5,001) RATIO
0902 WRITE(6,0152)
      READ(5,0001,END=9791) DC
0904 WRITE(6,0157)
      READ(5,0001,END=9791) NN
      IF(LNOR.EQ.IRES) GO TO 0908
      WRITE(6,0144)
      WRITE(6,0145)
      READ(5,0151,END=9791) NORM
0908 WRITE(6,0137)
      READ(5,0001,END=9791)((SIGMA(I,J),J=1,NL),I=1,NL)
      WRITE(6,0139)
      READ(5,0001,END=9791)(U(I),I=1,NL)
      GO TO 0915
0910 WRITE(6,0149)
      READ(5,0001,END=9791) NI4
      WRITE(6,0150)
      READ(5,0001,END=9791) ALPHA
      WRITE(6,0152)
      READ(5,0001,END=9791) DC
      WRITE(6,0161)
      READ(5,001) IDX
0915 CALL CMAT1
      WRITE(6,0159) KSU
      WRITE(6,0121) NI

```

```

WRITE(6,0123) NJ
WRITE(6,0125) NI4
WRITE(6,0116) NL
WRITE(6,0127) NN
WRITE(6,0114) ALPHA
WRITE(6,0120) DC
IF(LNOR.EQ.IRES) GO TO 916
GO TO 0917
0916 WRITE(6,0163) IDX
WRITE(6,0165) RATIO
GO TO 0930
0917 IF(NORM.NE.KNORM) GO TO 0920
WRITE(6,0146) IEUC
GO TO 0930
0920 WRITE(6,0146) ISUP
0930 CONTINUE
WRITE(6,0131)
DO 940 I=1,NL
WRITE(6,0141)(SIGMA(I,J),J=1,NL)
940 CONTINUE
WRITE(6,0133)
DO 950 I=1,NL
WRITE(6,0141)(CMAT(I,J),J=1,NL)
950 CONTINUE
WRITE(6,0128)
WRITE(6,0141)(U(I),I=1,NL)
WRITE(6,0143)
READ(5,0151) IZ
IF(IZ.NE.IRES) GO TO 0900
POWER=0.0
C
C ** COMPUTE THE CRITICAL VALUE OF THE TEST STATISTIC **
C
CALL CRIT(NI,NJ,NI4,NL,ALPHA,CHIN,CHI,CRITV,ERROR)
IF(ERROR.EQ.1) GO TO 9795
C
C ** LOOP ON REPLICATION FOR THIS NORM **
C
DO 8500 I2=1,NN
IF(LNOR.EQ.IRES) GO TO 0990
CALL ORDER(NL,UNIF)
IF(NORM.NE.KNORM) GO TO 0970
CALL ASGNOR(DC,DCOM,UNIF)
GO TO 0990
0970 CALL ASGMAX(DC,DCOM,UNIF)
C
C ** LOOP ON ITERATIONS **
C
C
0990 IF(LNOR.NE.IRES) GO TO 1020
DO 1000 LL=1,NL
IF(LL.EQ.IDX) GO TO 0995
DCOM(LL)=UNIF(A)*DC/RATIO
GO TO 1000
0995 DCOM(LL)=DC
1000 CONTINUE
1020 DO 1050 I3=1,NL

```

```

      CALL ORDER(NI,UNIF)
      CALL FACOM(DCOM(I3))
      DO 1030 III=1,NL
      DO 1029 JJJ=1,NI
      A(JJJ,III)=0.
1029  CONTINUE
1030  CONTINUE
      DO 1040 KC=1,NI
      JR=KORD(KC)
      A(JR,I3)=FAC(KC)
1040  CONTINUE
1050  CONTINUE
C
C  **  GENERATE THE OBSERVATIONS  **
C
      DO 1500 II=1,NI
      DO 1490 JJ=1,NJ
      DO 1480 KK=1,NI4
      CALL XVEC1(RNORM1,UNIF)
      DO 1470 LL=1,NL
      Y(II,JJ,KK,LL)=A(II,LL)+XVEC(LL)
1470  CONTINUE
1480  CONTINUE
1490  CONTINUE
1500  CONTINUE
C
C  **  COMPUTE THE MANOVA  **
C
C  **  COMPUTE THE CELL MEANS  **
C
      DO 1600 IC=1,NI
      DO 1590 JC=1,NJ
      DO 1580 LC=1,NL
      SUM=0.0
      DO 1570 KC=1,NI4
      SUM=SUM+Y(IC,JC,KC,LC)
1570  CONTINUE
      C(IC,JC,LC)=SUM
1580  CONTINUE
1590  CONTINUE
1600  CONTINUE
C
C  **  COMPUTE THE COLUMN TREATMENTS  **
C
      DO 1700 JC=1,NJ
      DO 1690 LC=1,NL
      SUM=0.0
      DO 1680 IC=1,NI
      DO 1670 KC=1,NI4
      SUM=SUM+Y(IC,JC,KC,LC)
1670  CONTINUE
1680  CONTINUE
      T(JC,LC)=SUM
1690  CONTINUE
1700  CONTINUE
C
C  **  COMPUTE THE ROW TREATMENTS  **

```



```

C
DO 1800 IC=1,NI
DO 1790 LC=1,NL
SUM=0.0
DO 1780 JC=1,NJ
DO 1770 KC=1,NI4
SUM=SUM+Y(IC,JC,KC,LC)
1770 CONTINUE
1780 CONTINUE
R(IC,LC)=SUM
1790 CONTINUE
1800 CONTINUE
C
C ** COMPUTE THE GRAND TOTALS **
C
DO 1900 LC=1,NL
SUM=0.0
DO 1890 KC=1,NI4
DO 1880 JC=1,NJ
DO 1870 IC=1,NI
SUM=SUM+Y(IC,JC,KC,LC)
1870 CONTINUE
1880 CONTINUE
1890 CONTINUE
G(LC)=SUM
1900 CONTINUE
C
C ** COMPUTE THE H1 MATRIX **
C
DO 2000 IL=1,NL
DO 1990 JL=1,NL
SUM=0.0
DO 1980 IC=1,NI
SUM=SUM+R(IC,IL)*R(IC,JL)
1980 CONTINUE
NJK=NJ*NI4
Y1=NJK
SUM=SUM/Y1
NIJK=NI*NJ*NI4
Y1=NIJK
H1(IL,JL)=SUM-G(IL)*G(JL)/Y1
1990 CONTINUE
2000 CONTINUE
C
C ** COMPUTE THE E MATRIX **
C
DO 2100 IL=1,NL
DO 2090 JL=1,NL
SUM1=0.0
DO 2060 IC=1,NI
DO 2050 JC=1,NJ
DO 2040 KC=1,NI4
SUM1=SUM1+Y(IC,JC,KC,IL)*Y(IC,JC,KC,JL)
2040 CONTINUE
2050 CONTINUE
2060 CONTINUE
SUM2=0.0

```

```

      DO 2080 IC=1,NI
      DO 2070 JC=1,NJ
      SUM2=SUM2+C(IC,JC,IL)*C(IC,JC,JL)
2070  CONTINUE
2080  CONTINUE
      Y1=NI4
      E(IL,JL)=SUM1-SUM2/Y1
2090  CONTINUE
2100  CONTINUE
      IF(NI4.NE.1) GO TO 2600
C
C  ** COMPUTE THE H2 MATRIX **
C
      DO 2200 IL=1,NL
      DO 2190 JL=1,NL
      SUM=0.0
      DO 2180 JC=1,NJ
      SUM=SUM + T(JC,IL)*T(JC,JL)
2180  CONTINUE
      Y1=(NI*NI4)
      SUM=SUM/Y1
      Y1=(NI*NJ*NI4)
      H2(IL,JL)=SUM - G(IL)*G(JL)/Y1
2190  CONTINUE
2200  CONTINUE
C
C  ** COMPUTE THE TOTALS MATRIX **
C
      DO 2300 IL=1,NL
      DO 2290 JL=1,NL
      SUM1=0.0
      DO 2260 IC=1,NI
      DO 2250 JC=1,NJ
      DO 2240 KC=1,NI4
      SUM1=SUM1 + Y(IC,JC,KC,IL)*Y(IC,JC,KC,JL)
2240  CONTINUE
2250  CONTINUE
2260  CONTINUE
      Y1=(NI*NJ*NI4)
      Z(IL,JL) = SUM1 - G(IL)*G(JL)/Y1
2290  CONTINUE
2300  CONTINUE
C
C  ** COMPUTE THE H3 MATRIX **
C
      DO 2400 IL=1,NL
      DO 2390 JL=1,NL
      H3(IL,JL)=Z(IL,JL)-H1(IL,JL)-H2(IL,JL)-E(IL,JL)
2390  CONTINUE
2400  CONTINUE
C
C  ** REPLACE E MATRIX WITH H3 MATRIX **
C
      DO 2500 IL=1,NL
      DO 2490 JL=1,NL
      E(IL,JL)=H3(IL,JL)
2490  CONTINUE

```

```

2500  CONTINUE
C
C  **  COMPUTE THE TEST STATISTIC OF THE MANOVA  **
C
2600  CALL MATADD
      V(1)=2.
      CALL GJR(E,20,20,NL,NL,$9793,JD,V)
      IF(V(2).EQ.0.0) GO TO 9793
      ED=V(2)
      ES=V(1)
      V(1)=2.
      CALL GJR(HD,20,20,NL,NL,$9793,JD,V)
      IF(V(2).EQ.0.0) GO TO 9793
      HT=V(2)
      HS=V(1)
      ED=EXP(ED)
      ED=SIGN(ED,ES)
      HT=EXP(HT)
      HT=SIGN(HT,HS)
      CV=ED/HT
C
C  **  TEST THE CRITICAL VALUE OF THE TEST STATISTIC  **
C
      IF(CV.GT.CRITV) GO TO 3000
      POWER=POWER + 1.0
3000  CONTINUE
8500  CONTINUE
      GO TO 9801
C
C  **  ERROR MESSAGES  **
C
9791  WRITE(6,0101)
      GO TO 9801
9793  WRITE(6,0105)
      GO TO 9801
9795  WRITE(6,0103)
      GO TO 9801
C
C  **  COMPUTE THE POWERS BY SAMPLE SIZE  **
C
9801  W=NN
      POWER=POWER/W
C
C  **  OUTPUT SECTION  **
C
      WRITE(6,0118)
      WRITE(6,0122)
      WRITE(6,0124) NI4,POWER
      WRITE(6,0147)
      READ(5,0151,END=9791) IZ
      IF(IZ.NE.IRES) GO TO 9990
      WRITE(6,0148)
      READ(5,0151,END=9791) IZ
      IF(IZ.NE.IRES) GO TO 0900
      GO TO 0910
9990  CONTINUE
      END

```

```

SUBROUTINE CRIT(NI,NJ,I4,NL,ALPHA,CHIN,CHI,CRITV,ERROR)
C ** THIS SUBROUTINE COMPUTES THE SECOND-ORDER APPROXIMATION
C ** OF THE CRITICAL VALUE OF THE MANOVA TEST USING THE BOX
C ** METHOD BY MEANS OF A NONLINEAR SEARCH OPTIMIZATION
C ** ROUTINE FOR A GIVEN PROBABILITY OF TYPE I ERROR, ALPHA.
  INTEGER ERROR
  S=1. - ALPHA
  KOUNT=1
  DEL=.01
  P=NL
  Q1=(NI-1)
  SN=(NI*NJ*(I4-1))
  IF(I4.EQ.1) SN=(NI-1)*(NJ-1)
  ZR=NI
  BN=SN+ZR
  Q2=ZR-Q1
  CM=BN-Q2-.5*(P+Q1+1.)
  G=(P*Q1*(P**2+Q1**2-5.))/48.
  KDF1=NL*(NI-1)
  KDF2=KDF1+4
  X=CHIN(ALPHA,KDF1,$900)
100  Z=CHI(X,KDF1,$900)+(CHI(X,KDF2,$900)-CHI(X,KDF1,$900))*G/(CM**2)
  XEWF=S-Z
  IF(XEWF.LT.0.0) XEWF=-XEWF
  IF(KOUNT.NE.1) GO TO 200
  KOUNT=KOUNT+1
150  Y=X
  X=Y+DEL
  OLDF=XEWF
  GO TO 100
200  IF(XEWF-OLDF)201,201,202
201  IF(XEWF.LT..00001) GO TO 800
  DEL=DEL*3.0
  OLDF=XEWF
  GO TO 150
202  DEL=DEL*(-.5)
  X=Y
  XEWF=OLDF
  GO TO 150
800  CRITV=EXP(X/(-CM))
  GO TO 950
900  ERROR=1
  GO TO 990
950  ERROR=2
990  CONTINUE
  RETURN
  END

```

```

SUBROUTINE CMAT1
C ** THIS SUBROUTINE COMPUTES THE C-MATRIX REQUIRED TO GENERATE
C ** MULTIVARIATE NORMAL RANDOM VECTORS, SUCH THAT CC'=SIGMA,
C ** WHERE SIGMA IS THE POPULATION COVARIANCE MATRIX.

```

```

COMMON /TWO/ N
COMMON /FIVE/ SIGMA(20,20)
COMMON /SIX/ CMAT(20,20)
DO 110 J=1,N
  IF(J.GE.2) GO TO 91
  DO 81 I=1,N
    CMAT(I,1)=SIGMA(I,1)/SQRT(SIGMA(1,1))
81  CONTINUE
  GO TO 110
91  DO 105 I=1,N
    IF(J.GE.I+1) GO TO 104
    IF(J.LE.1) GO TO 95
    SUB1=0.0
    L=I-1
    DO 93 K=1,L
      SUB1=SUB1+CMAT(I,K)**2
93  CONTINUE
    CMAT(I,J)=SQRT(SIGMA(I,J)-SUB1)
    GO TO 105
95  SUB2=0.0
    L=J-1
    DO 97 K=1,L
      SUB2=SUB2+CMAT(I,K)*CMAT(J,K)
97  CONTINUE
    CMAT(I,J)=(SIGMA(I,J)-SUB2)/CMAT(J,J)
    GO TO 105
104 CMAT(I,J)=0.0
105 CONTINUE
110 CONTINUE
    RETURN
    END

```

```

SUBROUTINE FACOM(D)
COMMON /NINE/ X(20)
COMMON /ELEVEN/ NI
DO 50 I=1,NI
  X(I)=0.
50  CONTINUE
  IMOD=0
  Y=NI
  NN=NI
  IF(AHOD(Y,2.).GT..1) IMOD=1
  IF(IMOD.EQ.1) NN=NN-1
  Y=NN
  R=D/Y
  K=SQRT(R)
  DO 100 I=1,NN,2
    X(I)=R
    I=I+1
    X(I)=-R
100 CONTINUE
  IF(IMOD.EQ.1) X(NI)=0.
  RETURN
  END

```

```

SUBROUTINE XVEC1(RNORM1,UNIF)
C ** THIS SUBROUTINE GENERATES MULTIVARIATE NORMAL RANDOM VECTORS
C ** USING THE TRANSFORMATION  $Y=CX + U$ , WHERE C IS THE MATRIX
C ** FROM SUBROUTINE CMAT, AND X IS A P-DIMENSIONAL VECTOR FROM
C **  $N(0,1)$ .
COMMON /TWO/ N
COMMON /SIX/ CMAT(20,20)
COMMON /SEVEN/ ZVEC(20),U(20),XVEC(20),BUF(20)
DO 27 I=1,N,2
ZVEC(I)=RNORM1(UNIF,RNORM2,0.0,1.0)
I1=I+1
ZVEC(I1)=RNORM2
27 CONTINUE
DO 121 I=1,N
SUM=0.0
DO 111 J=1,N
SUM=SUM+CMAT(I,J)*ZVEC(J)
111 CONTINUE
BUF(I)=SUM
121 CONTINUE
DO 131 K=1,N
XVEC(K)=BUF(K)+U(K)
131 CONTINUE
RETURN
END

```

```

SUBROUTINE ORDER(N,UNIF)
C ** THIS SUBROUTINE RANDOMLY ASSIGNS ORDER TO A P-DIMENSIONAL
C ** VECTOR'S COMPONENTS.
COMMON /THREE/ KORD(20)
COMMON /FOUR/ KUSED(20)
DO 100 I=1,N
KORD(I)=0
KUSED(I)=0
100 CONTINUE
LEFT=N
DO 500 J=1,N
X=UNIF(X)
DO 400 K=1,LEFT
Y=FLOAT(K)
Y=Y/FLOAT(LEFT)
IF(X.GT.Y) GO TO 400
LU=0
DO 300 M=1,N
IF(KUSED(M).NE.0) GO TO 300
LU=LU+1
IF(LU.NE.K) GO TO 300
KUSED(M)=1
KORD(J)=M
GO TO 450
300 CONTINUE
400 CONTINUE
450 LEFT=LEFT-1
500 CONTINUE
RETURN
END

```

```

      FUNCTION RNORM1(UNIF,RNORM2,U,SIG2)
C  ** THIS FUNCTION PRODUCES INDEPENDENT NORMAL VARIATES WITH MEAN
C  ** U AND VARIANCE SIG2 BY MEANS OF THE BOX AND MULLER
C  ** TRANSFORMATION OF UNIFORM(0,1) DEVIATES.
      TPI=6.2831852
      A=UNIF(X)
      B=UNIF(X)
      RNORM1=U+SQRT(-2.0*SIG2*ALOG(A))*COS(TPI*B)
      RNORM2=U+SQRT(-2.0*SIG2*ALOG(A))*SIN(TPI*B)
      RETURN
      END

      SUBROUTINE ASGNOR(D,DCOM,UNIF)
C  ** THIS SUBROUTINE RANDOMLY ASSIGNS THE COMPONENTS OF A EUCLIDEAN
C  ** NORM SUCH THAT THE COMPONENTS ARE IN NORM EQUAL TO THE ORIGINAL
C  ** NORM.
      COMMON /TWO/ N
      COMMON /THREE/ KORD(20)
      DIMENSION DCOM(20)
      R=D**2
      M=N-1
      DO 100 I=1,M
        J=KORD(I)
        DCOM(J)=R*UNIF(A)
        R=R-DCOM(J)
100    CONTINUE
        J=KORD(N)
        DCOM(J)=R
      DO 200 K=1,N
        DCOM(K)= SQRT(DCOM(K))
200    CONTINUE
      RETURN
      END

      SUBROUTINE ASGMAX(D,DCOM,UNIF)
C  ** THIS SUBROUTINE RANDOMLY ASSIGNS THE COMPONENTS OF A SUPREMUM
C  ** NORM SUCH THAT THE COMPONENTS ARE IN NORM EQUAL TO THE ORIGINAL
C  ** NORM.
      COMMON /TWO/ N
      COMMON /THREE/ KORD(20)
      DIMENSION DCOM(20)
      M=N-1
      DO 100 I=1,M
        J=KORD(I)
        DCOM(J)=D*UNIF(A)
100    CONTINUE
        J=KORD(N)
        DCOM(J)=D
      RETURN
      END

```

```
      FUNCTION UNIF(A)
C
C  ** THIS FUNCTION PRODUCES DEVIATES WITH UNIFORM(0,1) DISTRIBUTION
C  ** BY THE MULTIPLICATIVE CONGRUENTIAL OVERFLOW METHOD. IT WILL
C  ** WORK ONLY ON A COMPUTER WITH A 36 BIT WORD SIZE.
C
      DATA IY/96581/
      IY=IY*3125
      IF(IY) 5,6,6
5     IY=IY+1+34359738367
6     YFL=IY
      UNIF=YFL*2.0**(-35)
      RETURN
      END
```


APPENDIX B

This appendix contains a complete FORTRAN IV listing of a program which computes the multiple correlation coefficients of a set of responses, given the sample correlation or covariance matrix. The program is interactive; input is in free-field format. The program listing follows an example of its use.

@XQT ZCORR.MLT

*** MULTIPLE CORRELATION COEFFICIENT PROGRAM ***

ENTER THE DIMENSION OF THE RESPONSE

7

ENTER THE SAMPLE COVARIANCE MATRIX

1.,.00,-.06,-.12,.00,-.17,.16
 .00,1.,.01,-.11,.01,-.04,.76
 -.06,.01,1.,.68,-.49,.56,.07
 -.12,-.11,.68,1.,-.21,.72,-.04
 .00,.01,-.49,-.21,1.,-.26,-.11
 -.17,-.04,.56,.72,-.26,1.,-.08
 .16,.76,.07,-.04,-.11,-.08,1.

R(1)**2 = .086375

R(1) = .293896

R(2)**2 = .611968

R(2) = .782284

R(3)**2 = .594059

R(3) = .770752

R(4)**2 = .657847

R(4) = .811078

R(5)**2 = .496048

R(5) = .704307

R(6)**2 = .553953

R(6) = .744280

R(7)**2 = .622111

R(7) = .788740

DETERMINANT IS .06612

NORMAL EXIT. EXECUTION TIME: 166 MLSEC.

```

      INTEGER POS(20)
      DIMENSION S(20,20),C(20,20),S12(20,20),S12T(20,20),S22(20,20)
      DIMENSION JD(20),V(2),A(20,20),B(20,20)
001  FORMAT( )
      WRITE(6,101)
101  FORMAT(/,5X,'*** MULTIPLE CORRELATION COEFFICIENT PROGRAM ***')
      WRITE(6,103)
103  FORMAT(/,2X,'ENTER THE DIMENSION OF THE RESPONSE')
      READ(5,001) NL
      WRITE(6,105)
105  FORMAT(/,2X,'ENTER THE SAMPLE COVARIANCE MATRIX')
      READ(5,001) ((C(I,J),J=1,NL),I=1,NL)
      N2=NL-1
      N1=1
      DO 900 IP=1,NL
        IF(IP.NE.1) GO TO 175
        DO 150 IC=1,NL
          POS(IC)=IC
150      CONTINUE
        GO TO 200
175      POS(1)=IP
          PCS(2)=1
          IK=2
          DO 190 IC=3,NL
            IF(IP.EQ.IK) IK=IK+1
            POS(IC)=IK
            IK=IK+1
190      CONTINUE
200      CONTINUE
          DO 250 IC=1,NL
            IA=POS(IC)
            DO 240 JC=1,NL
              JA=POS(JC)
              S(IC,JC)=C(IA,JA)
240      CONTINUE
250      CONTINUE
          DO 300 IC=1,N2
            IA=IC+1
            DO 290 JC=1,N2
              JA=JC+1
              S22(IC,JC)=S(IA,JA)
290      CONTINUE
300      CONTINUE
          DO 320 JC=1,N2
            JA=JC+1
            S12T(1,JC)=S(1,JA)
320      CONTINUE
          DO 340 IC=1,N2
            IA=IC+1
            S12(IC,1)=S(IA,1)
340      CONTINUE
          V(1)=1.
          CALL GJR(S22,20,20,N2,N2,$950,JD,V)
          CALL MXMLT(S12T,S22,A,N1,N2,N2,20,20)
          CALL MXMLT(A,S12,B,N1,N2,N1,20,20)
          R=B(1,1)/S(1,1)

```

```
      WRITE(6,109) IP,R
109   FORMAT(//,2X,'R(',I1,')**2 =',F10.6)
      R=SQRT(R)
      WRITE(6,111) IP,R
111   FORMAT(//,2X,'R(',I1,')    =',F10.6)
900   CONTINUE
      V(1)=2.
      CALL GJR(C,20,20,NL,NL,$950,JD,V)
      D=EXP(V(2))
      D=SIGN(D,V(1))
      WRITE(6,113) D
113   FORMAT(//,2X,'DETERMINANT IS ',F10.5)
950   CONTINUE
      END
```

APPENDIX C

This appendix contains a complete FORTRAN IV listing of a program which computes the test statistic used to test whether two sets of responses are independent using the results of (2.39) and (2.40). The program is interactive; input is in free-field format. The listing follows an example of its use.

@XQT ZCORR.INDSET

TEST FOR INDEPENDENCE OF 2 SETS OF VARIATES

ENTER THE DIMENSION OF THE RESPONSE

6

ENTER THE NUMBER OF VARIATES IN 1ST SET

2

ENTER THE NUMBER OF VARIATES IN 2ND SET

4

ENTER THE SAMPLE COVARIANCE MATRIX

1.,.00,-.11,.01,-.04,.76
 .01,1.,.68,-.49,.56,.07
 -.11,.68,1.,-.21,.72,-.04
 .01,-.49,-.21,1.,-.26,-.11
 -.04,.56,.72,-.26,1.,-.08
 .76,.067,-.04,-.11,-.08,1.

ENTER THE INDEX NRS OF 1ST SET OF VARIATES

1,6

ENTER THE INDEX NRS OF 2ND SET OF VARIATES

2,3,4,5

ENTER THE SAMPLE SIZE

42

ENTER ALPHA

.05

** DIMENSION OF THE RESPONSE = 6

** NR OF VARIATES IN 1ST SET = 2

** NR OF VARIATES IN 2ND SET = 4

** SAMPLE SIZE = 42

** ALPHA = .05

** REARRANGED COVARIANCE MATRIX **

1.0000	.7600	.0100	-.1100	.0100	-.0400
.7600	1.0000	.0700	-.0400	-.1100	-.0800
.0100	.0700	1.0000	.6800	-.4900	.5600
-.1100	-.0400	.6800	1.0000	-.2100	.7200
.0100	-.1100	-.4900	-.2100	1.0000	-.2600
-.0400	-.0800	.5600	.7200	-.2600	1.0000

** TEST STATISTIC = 3.5587
** CRITICAL VALUE = 15.5073
** FAIL TO REJECT INDEPENDENCE **

NORMAL EXIT. EXECUTION TIME: 190 MLSEC.

```

      INTEGER POS(20)
      DIMENSION R11(20,20),R22(20,20)
      DIMENSION R(20,20),C(20,20),JD(20),V(2)
001  FORMAT( )
      WRITE(6,101)
101  FORMAT(1H1,2X,'**TEST FOR INDEPENDENCE OF 2 SETS OF VARIATES**')
      WRITE(6,103)
103  FORMAT(/,2X,'ENTER THE DIMENSION OF THE RESPONSE')
      READ(5,001,END=995) NL
      WRITE(6,105)
105  FORMAT(/,2X,'ENTER THE NUMBER OF VARIATES IN 1ST SET')
      READ(5,001,END=995) N1
      WRITE(6,107)
107  FORMAT(/,2X,'ENTER THE NUMBER OF VARIATES IN 2ND SET')
      READ(5,001,END=995) N2
      WRITE(6,109)
109  FORMAT(/,2X,'ENTER THE SAMPLE COVARIANCE MATRIX')
      READ(5,001,END=995)((R(I,J),J=1,NL),I=1,NL)
      WRITE(6,111)
111  FORMAT(/,2X,'ENTER THE INDEX NRS OF 1ST SET OF VARIATES')
      READ(5,001,END=995)(POS(I),I=1,N1)
      WRITE(6,113)
113  FORMAT(/,2X,'ENTER THE INDEX NRS OF 2ND SET OF VARIATES')
      N3=N1+1
      READ(5,001,END=995)(POS(I),I=N3,NL)
      WRITE(6,115)
115  FORMAT(/,2X,'ENTER THE SAMPLE SIZE')
      READ(5,001,END=995) NS
      WRITE(6,117)
117  FORMAT(/,2X,'ENTER ALPHA')
      READ(5,001,END=995) ALPHA
      DO 300 IC=1,NL
        IA=POS(IC)
        DO 290 JC=1,NL
          JA=POS(JC)
          C(IC,JC)=R(IA,JA)
290  CONTINUE
300  CONTINUE
      WRITE(6,121) NL
121  FORMAT(/,5X,'**  DIMENSION OF THE RESPONSE =',I2)
      WRITE(6,122) N1
122  FORMAT(/,5X,'**  NR OF VARIATES IN 1ST SET =',I2)
      WRITE(6,123) N2
123  FORMAT(/,5X,'**  NR OF VARIATES IN 2ND SET =',I2)
      WRITE(6,124) NS
124  FORMAT(/,5X,'**  SAMPLE SIZE =',I3)
      WRITE(6,125) ALPHA
125  FORMAT(/,5X,'**  ALPHA =',F3.2)
      WRITE(6,126)
126  FORMAT(/,5X,'**  REARRANGED COVARIANCE MATRIX  **')
127  FORMAT(/,2X,8(1X,F8.4))
      DO 200 I=1,NL
        WRITE(6,127)(C(I,J),J=1,NL)
200  CONTINUE
      DO 400 IC=1,N1
        DO 390 JC=1,N1

```



```

R11(IC,JC)=C(IC,JC)
390 CONTINUE
400 CONTINUE
DO 500 IC=1,N2
  IA=N1+IC
DO 490 JC=1,N2
  JA=N1+JC
  R22(IC,JC)=C(IA,JA)
490 CONTINUE
500 CONTINUE
YN1=N1
YNS=NS
YN2=N2
V1=YNS-(YN1+YN2+1.)/2.
V(1)=2.
CALL GJR(C,20,20,NL,NL,$995,JD,V)
ED=EXP(V(2))
T=SIGN(ED,V(1))
V(1)=2.
CALL GJR(R11,20,20,N1,N1,$995,JD,V)
ED=EXP(V(2))
B1=SIGN(ED,V(1))
V(1)=2.
CALL GJR(R22,20,20,N2,N2,$995,JD,V)
ED=EXP(V(2))
B2=SIGN(ED,V(1))
CVT=-ALOG(T/(B1*B2))*V1
IDF=N1*N2
CRIT=CHIN(ALPHA,IDF,$995)
WRITE(6,131) CVT
131 FORMAT(/,5X,'** TEST STATISTIC =',F10.4)
WRITE(6,133) CRIT
133 FORMAT(/,5X,'** CRITICAL VALUE =',F10.4)
IF(CVT.GT,CRIT) GO TO 800
WRITE(6,135)
135 FORMAT(/,5X,'** HENCE FAIL TO REJECT INDEPENDENCE **')
GO TO 900
800 WRITE(6,137)
137 FORMAT(/,5X,'** HENCE REJECT INDEPENDENCE **')
900 CONTINUE
995 CONTINUE
END

```

APPENDIX D

This appendix contains a complete FORTRAN IV listing of a program used to test whether a set of responses is independent using the results of (2.38). The program is interactive; input is in free-field format. The listing follows an example of its use.

@XQT ZCORR.INDEP

** TEST FOR COMPLETE INDEPENDENCE **

ENTER THE DIMENSION OF THE RESPONSE

4

ENTER THE SAMPLE CORRELATION MATRIX

1.,.68,-.49,.56
 .68,1.,-.21,.72
 -.49,-.21,1.,-.26
 .56,.72,-.26,1.

ENTER THE SAMPLE SIZE

42

ENTER ALPHA

.05

** DIMENSION OF THE RESPONSE = 4

** SAMPLE SIZE = 42

** ALPHA = .050

** CORRELATION MATRIX **

1.0000	.6800	-.4900	.5600
.6800	1.0000	-.2100	.7200
-.4900	-.2100	1.0000	-.2600
.5600	.7200	-.2600	1.0000

THE VALUE OF THE TEST STATISTIC = 62.03133

THE CRITICAL VALUE = 12.59159

** HENCE REJECT INDEPENDENCE **

NORMAL EXIT. EXECUTION TIME: 109 MLSEC.

```

        DIMENSION R(20,20),J(20),V(2)
        WRITE(6,101)
101    FORMAT(1H1,5X,'** TEST FOR COMPLETE INDEPENDENCE **')
        WRITE(6,103)
103    FORMAT(/,2X,'ENTER DIMENSION OF THE RESPONSE')
        READ(5,001,END=999) NL
001    FORMAT( )
        WRITE(6,105)
105    FORMAT(/,2X,'ENTER THE SAMPLE CORRELATION MATRIX')
        READ(5,001,END=999)((R(I,J),J=1,NL),I=1,NL)
        WRITE(6,107)
107    FORMAT(/,2X,'ENTER THE SAMPLE SIZE')
        READ(5,001,END=999) NK
        WRITE(6,109)
109    FORMAT(/,2X,'ENTER ALPHA')
        READ(5,001,END=999) ALPHA
        WRITE(6,121) NL
121    FORMAT(/,5X,'** DIMENSION OF THE RESPONSE =',I2)
        WRITE(6,125) NK
125    FORMAT(/,5X,'** SAMPLE SIZE =',I4)
        WRITE(6,127) ALPHA
127    FORMAT(/,5X,'** ALPHA =',F4.3)
        WRITE(6,122)
122    FORMAT(/,5X,'** CORRELATION MATRIX **')
        DO 200 I=1,NL
            WRITE(6,123)(R(I,J),J=1,NL)
123    FORMAT(/,2X,10(1X,F6.4))
200    CONTINUE
        V(1)=2.
        CALL GJR(R,20,20,NL,NL,$900,JC,V)
        ED=V(2)
        YN=NK-1
        YM=(2*NL+5)
        YM=YM/6.
        YN=-(YN-YM)
        CHISQ=YN*ED
        IDF=(NL*(NL-1))/2
        CV=CHIN(ALPHA,IDF,$900)
        WRITE(6,111) CHISQ
111    FORMAT(/,5X,'THE VALUE OF THE TEST STATISTIC =',F10.5)
        WRITE(6,113) CV
113    FORMAT(/,5X,'THE CRITICAL VALUE =',F10.5)
        IF(CHISQ.GE.CV) GO TO 800
        WRITE(6,115)
115    FORMAT(/,5X,'** HENCE FAIL TO REJECT INDEPENDENCE **')
        GO TO 900
800    WRITE(6,117)
117    FORMAT(/,5X,'** HENCE REJECT INDEPENDENCE **')
900    CONTINUE
999    CONTINUE
        END

```

BIBLIOGRAPHY

1. Afifi, A. A. and Azen, S. P., Statistical Analysis, A Computer Oriented Approach, Academic Press, New York, 1972.
2. Anderson, T. W., An Introduction to Multivariate Statistical Analysis, John Wiley and Sons, Inc., New York, 1958.
3. Andrews, D. F., "A Note of the Selection of Data Transformations," Biometrika, Vol. 58, 1971, No. 2, pp. 249.
4. Andrews, D. F., Gnanadesikan, R. and Warner, J. L., "Transformations of Multivariate Data," Biometrics, Vol. 27, 1971, pp. 825-840.
5. Bolch, B. W. and Huang, C. J. Multivariate Statistical Methods for Business and Economics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.
6. AR 10-4, dated 30 December 1974, Organization and Functions, U. S. Army Operational Test and Evaluation Agency, Headquarters, Department of the Army, Washington, D.C.
7. AR 71-3, dated 30 November 1974, Research and Development and Force Development Testing, Headquarters, Department of the Army, Washington, D. C.
8. AR 1000-1, dated 5 November 1974, Basic Policies for Systems Acquisition by the Department of the Army, Headquarters, Department of the Army, Washington, D.C.
9. Bartlett, S., "A Note on Multiplying Factors for Various Chi-Squared Approximations," Journal of the Royal Statistical Society, Series B, Vol. 16, 1954, pp. 296-298.
10. Bazaraa, M. S., An Efficient Cyclic Coordinate Method for Optimizing Penalty Functions, Unpublished Report, Georgia Institute of Technology, 1975.
11. Box, G. E. P., "A General Distribution Theory for a Class of Likelihood Criteria," Biometrika, Vol. 36, 1949, pp. 317-346.

12. Box, G. E. P., "Non-Normality and Tests on Variances," Biometrika, Vol. 40, 1953, pp. 318-335.
13. Box, G. E. P., "Some Theorems on Quadratic Forms Applied in the Study of Analysis of Variance Problems: II. Effect of Inequality of Variance and of Correlation of Errors in Two-way Classification," Annals Math. Stat., Vol. 25, 1954, pp. 484-498.
14. Box, G. E. P. and Muller, M. E., "A Note on the Generation of Random Normal Deviates," Ann Math. Stat., Vol. 29, 1958, pp. 610-611.
15. Box, G. E. P. and Cox, D. R., "An Analysis of Transformations," J. R. Statist. Soc. B, Vol. 26, 1964, pp. 211-252.
16. Cornfield, J. and Tukey, J. W., "Average Values of Mean Squares in Factorials," Annals of Math. Stat., Vol. 27, 1956, pp. 907-949.
17. Das Gupta, S. and Perlman, M. D., "On the Power of Wilks' U-Test for MANOVA," Journal of Multivariate Analysis, Vol. 3, 1973, pp. 220-225.
18. Dempster, A. P., Elements of Continuous Multivariate Analysis, Addison-Wesley Publishing Co., Reading, Mass., 1969.
19. Dempster, A. P., "An Overview of Multivariate Data Analysis," Journal of Multivariate Analysis, Vol. 1971, pp. 316-346.
20. Dixon, W. J., BMD Biomedical Computer Programs, 3rd Edition, University of California Press, Berkeley, 1974.
21. DoD Directive 5000.1, dated 1 September 1974, Acquisition of Major Defense Systems, Headquarters, Department of Defense, Washington, D. C.
22. DoD Directive 5000.2, dated 20 February 1974, The Decision Coordinating Paper (DCP) and the Defense Systems Acquisition Review Council (DSARC), Headquarters, Department of Defense, Washington, D. C.
23. Dolby, J. L., "On a Quick Method of Choosing a Transformation," Technometrics, Vol. 5, 1963, pp. 317-325.

24. Finn, J. D., A General Model for Multivariate Analysis, Holt, Rinehart and Winston, Inc., New York, 1974.
25. Fisher, R. A., "Frequency Distribution of the Values of the Correlation Coefficient in Samples from an Indefinitely Large Population," Biometrika, Vol. 10, 1915, pp. 507-521.
26. Fishman, G. S., Concepts and Methods in Discrete Event Digital Simulation, John Wiley & Sons, New York, 1973.
27. Gabriel, K. R., "Simultaneous Test Procedures in Multivariate Analysis of Variance," Biometrika, Vol. 55, 1968, pp. 489-504.
28. Gnanadesikan, R. and Wilk, M. B., "Data Analytic Methods in Multivariate Statistical Analysis," Multivariate Analysis II, P. R. Krishnaian, editor, pp. 593-638, Academic Press, New York, 1969.
29. Gnanadesikan, R. and Kettenring, J. R., "Robust Estimates, Residuals, and Outlier Detection with Multi-response Data," Biometrics, Vol. 28, 1972, pp. 81-124.
30. Graybill, F. A., An Introduction to Linear Statistical Models, Vol. I, McGraw-Hill Book Co., New York, 1961.
31. Heck, D. L., "Charts of Some Upper Percentage Points of the Distribution of the Largest Characteristic Root," Annals of Math. Stat., Vol. 31, 1960, pp. 625-642.
32. Himmelblau, D. M., Applied Nonlinear Programming, McGraw-Hill Book Co., New York, 1972.
33. Hines, W. W. and Montgomery, D. C., Probability and Statistics in Engineering and Management Science, The Ronald Press Company, New York, 1972.
34. Ito, K., "On the Effect of Heteroscedasticity and Nonnormality Upon some Multivariate Test Procedures," Multivariate Analysis--II, P. R. Krishnaian, editor, Academic Press, Inc., New York, 1969.
35. Kabe, D. G. and Gupta, R. P., Multivariate Statistical Inference, North-Holland Publishing Co., Amsterdam, 1973.
36. Kendall, M. G., A Course in Multivariate Analysis, Charles Griffin Co., Ltd., London, 1963.

37. Malkovich, J. F. and Afifi, A. A., "On Tests for Multivariate Normality," Journal of the American Statistical Association, Vol. 68, 1973, No. 341, pp. 176-179.
38. Miller, R. G., Simultaneous Statistical Inference, McGraw-Hill Book Co., New York, 1966.
39. Morrison, D. F., Multivariate Statistical Methods, McGraw-Hill Book Co., New York, 1967.
40. Nie, N., Bent, D. H., and Hull, C. H., Statistical Package for the Social Sciences, McGraw-Hill Book Co., New York, 1970.
41. Pearson, E. S., "The Analysis of Variance in Cases of Non-Normal Variation," Biometrika, Vol. 23, 1931, pp. 114-133.
42. Pearson, E. S. and Hartley, H. O., Biometrika Tables for Statisticians, Cambridge University Press, London, 1958.
43. Press, S. J., Applied Multivariate Analysis, Holt, Rinehart and Winston, Inc., New York, 1972.
44. Rao, C. R., "Some Characterizations of the Multivariate Normal Distribution," Multivariate Analysis II, P. R. Krishnaian, editor, Academic Press, Inc., New York, 1969.
45. Rao, C. R., Linear Statistical Inference and its Applications, 2nd Edition, John Wiley and Sons, New York, 1973.
46. Roy, S. N., Some Aspects of Multivariate Analysis, John Wiley & Sons, Inc., New York, 1957.
47. Roy, S. N. and Mikhail, W. F., "On the Monotonic Character of the Power Functions of Two Multivariate Tests," Ann. Math. Stat., Vol. 32, 1961, pp. 1145-1151.
48. Roy, S. N., Gnanadesikan, R., and Srivastava, J. N., Analysis and Design of Certain Quantitative Multi-response Experiments, Pergamon Press, New York, 1971.
49. Scheffe, Henry, The Analysis of Variance, John Wiley & Sons, Inc., New York, 1959.

50. Scheuer, E. and Stoller, D. S., "On the Generation of Normal Random Vectors," Technometrics, Vol. 4, 1962, pp. 278-281.
51. Tatsuka, M. M., Multivariate Analysis, John Wiley and Sons, Inc., New York, 1971.
52. Tiku, M. L., "Power Function of the F-Test Under Non-Normal Situations," Journal of the American Statistical Association, Vol. 56, 1971, pp. 913-916.
53. Tukey, J. W., "One Degree of Freedom for Non-additivity," Biometrics, Vol. 5, 1949, pp. 232-242.
54. Tukey, J. W., "On the Comparative Anatomy of Transformations," Ann. Math. Statist., Vol. 28, 1959, pp. 602-632.
55. U. S. Army Combat Developments Command, Integrated Battlefield Control System: Division System Definition, Washington, D. C., 1973.
56. Van de Geer, J. P., Introduction to Multivariate Analysis, W. H. Freeman and Co., San Francisco, 1971.
57. Wilks, S. S., "On the Independence of k Sets of Normally Distributed Statistical Variables," Econometrica, Vol. 3, 1935, pp. 309-326.